COMPACT COMPOSITION OPERATORS ON SOME MÖBIUS INVARIANT BANACH SPACES

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ABSTRACT

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Let B_p $(1 < p < \infty)$ be a Besov space and $\mathcal{B}$ the Bloch space. We give Carleson type measure characterizations for compact composition operators $C_\phi : B_p \to B_q$ $(1 < p \leq q < \infty)$, $C_\phi : B_p \to BMOA$, and $C_\phi : \mathcal{B} \to VMOA$. We show that if C_ϕ is bounded on some Besov space then C_ϕ is compact on larger Besov spaces if and only if it is compact on the Bloch space. Also, if ϕ is a boundedly valent holomorphic self-map of the unit disc U such that $\phi(U)$ lies inside a polygon inscribed in the unit circle, then C_ϕ is compact on $BMOA$, and on $VMOA$ if and only if it is compact on the Bloch space.

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Introduction

Let ϕ be a holomorphic self-map of the open unit disc U, H^2 the Hilbert space of functions holomorphic on U with square summable power series coefficients. Associate to ϕ the composition operator C_ϕ defined by

$$C_\phi f = f \circ \phi \,,$$

for f holomorphic on U. This is the first setting in which composition operators were studied. By Littlewood's Subordination Principle every composition operator takes H^2 into itself.

A natural question to ask is which composition operators on H^2 are compact. Shapiro in [31], using the Nevanlinna counting function, characterized the compact composition operators on H^2 as follows: C_ϕ is a compact operator on H^2 if and only if

$$\lim_{|w|\to 1} \frac{N_\phi(w)}{-\log|w|} = 0 \,.$$

A natural follow up question is about the boundedness and compactness of composition operators on other function spaces. We know the answer to this question in a variety of spaces.

MacCluer in [20], Madigan in [21], Roan in [25], and Shapiro in [30] have characterized the boundedness and compactness of C_ϕ in "small" spaces.

In "large" spaces, MacCluer and Shapiro show in [19] that C_ϕ is compact on Bergman spaces if and only if ϕ does not have an angular derivative at any point of

∂U. The angular derivative criterion is not sufficient, in general, in smaller spaces unless we put extra conditions on the symbol. For example they showed that it is sufficient on Hardy spaces, if the symbol is boundedly valent.

The *Bloch space* $\mathcal{B}$ is the space of holomorphic functions f on U such that $||f||_{\mathcal{B}} = \sup_{z\in U} |f'(z)|(1-|z|^2) < \infty$. It becomes a Banach space with norm $|f(0)| + ||f||_{\mathcal{B}}$. A linear subspace X of $\mathcal{B}$ with a seminorm $||.||_X$ is *Möbius invariant* if for all Möbius transformations ϕ and all $f \in X$, $f \circ \phi \in X$ and $||f \circ \phi||_X = ||f||_X$, and there exists a positive constant c such that $||f||_{\mathcal{B}} \leq ||f||_X$. It is easy to see that $\mathcal{B}$ is a Möbius invariant space.

A *Möbius invariant* Banach space X is a Möbius invariant subspace of the Bloch space with a seminorm $||.||_X$, whose norm is $f \to ||f||_X$ or $f \to |f(0)| + ||f||_X$. Rubel and Timoney showed in [26] that $\mathcal{B}$ is the largest Möbius invariant Banach space that possesses a decent linear functional. Other Möbius invariant Banach spaces include the Besov spaces, the space of holomorphic functions with bounded mean oscillation $BMOA$, and the space of holomorphic functions with vanishing mean oscillation $VMOA$. We will define and discuss properties of these spaces in chapter 1.

Madigan and Matheson show in [22] that C_ϕ is compact on the Bloch space if and only if

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0\,.$$

They also show that if C_ϕ is compact on $\mathcal{B}$ then it can not have an angular derivative at any point of ∂U.

In this thesis we study the compact composition operators on B_p $(1 < p < \infty)$, on $BMOA$, and on $VMOA$. For the rest of this introduction let X denote one of these spaces, unless otherwise stated. One way to approach this problem is to relate it to properties of ϕ. That is to see how fast or how often $\phi(U)$ touches ∂U. In every function space that compact composition operators have been studied, the first class of examples were provided by symbols ϕ such that $\phi(U)$ is a relatively compact subset of U. For the spaces that we study this is not an exception. Moreover, if C_ϕ is compact on X then C_ϕ can not have an angular derivative at any point ∂U since, if C_ϕ is compact on X then C_ϕ is compact on the Bloch space (see Proposition 3.2).

In Chapter 2, using counting functions, we give a Carleson measure characterization of compact operators $C_\phi : B_p \to B_q$ $(1 < p \le q < \infty)$ and $C_\phi : B_p \to BMOA$ $(1 < p \le 2)$. MacCluer and Shapiro give in [19] Carleson measure characterization of compact composition operators on the Dirichlet space $\mathcal{D}$, which is a Besov space ($p = 2$). Let α_λ $(\lambda \in U)$ be the basic conformal automorphism defined by $\alpha_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$. We prove the following theorems.

Theorem 2.7 *Let $1 < p \le q < \infty$. Then, the following are equivalent:*

1. *$C_\phi : B_p \to B_q$ is a compact operator.*
2. *$N_q(w, \phi)dA(w)$ is a vanishing q-Carleson measure.*
3. *$||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.*

Theorem 2.8 *The following are equivalent:*

1. *$C_\phi : \mathcal{D} \to BMOA$ is a compact operator.*
2. *$||C_\phi \alpha_\lambda||_* \to 0$, as $|\lambda| \to 1$.*

The main steps in the proof of the two theorems above are the following. First we characterize the vanishing p-Carleson measures (see Proposition 2.5). Then we give a general characterization of compact composition operators on certain Banach spaces of analytic functions in terms of bounded sequences that converge to 0 uniformly on compact subsets of U (see Lemma 2.10 and Lemma 2.11). Lastly a technique given by Arazy, Fisher, and Peetre in [2] and by Luecking in [17] and [18].

In Chapter 3 we first give another characterization of compact composition operators on the Bloch space. We prove the following theorem.

Theorem 3.1 *Let ϕ be a holomorphic self-map of U. Let $X = B_p$ $(1 < p < \infty)$, $BMOA$, or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is a compact operator if and only if*

$$\lim_{|\lambda| \to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0 \, .$$

Next we show that if $C_\phi : X \to X$ is compact then so is $C_\phi : \mathcal{B} \to \mathcal{B}$. Moreover we give conditions on the symbol under which the converse is valid as well. If X is a Besov space then the converse holds if we suppose that C_ϕ is bounded on a smaller Besov space. We prove the following theorem.

Theorem 3.7 *Let $1 < r < q$, $1 < p \leq q$, suppose that $C_\phi : B_r \to B_r$ is a bounded operator. Then the following are equivalent:*

1. *$C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

2. *$C_\phi : B_p \to B_q$ is a compact operator.*

3. *$C_\phi : \mathcal{D} \to BMOA$ is a compact operator.*

4. *$C_\phi : B_p \to BMOA$ is a compact operator.*

Next we describe the proof of the theorem above. At this point we have all the tools we need (see Lemma 2.11, Theorem 2.7, Theorem 2.8, and Theorem 3.1) to prove that *2, 3, 4*$\to$*1*. The hypothesis that $C_\phi : B_r \to B_r$ is a bounded operator is not needed for these implications. To prove the rest of the implications we first give a partial case. We show that if ϕ is a univalent function and C_ϕ is a compact operator on the Bloch space then $C_\phi : B_p \to B_q$ $(q > 2,\ 1 < p \leq q)$ is compact as well (see Theorem 3.5). Then we provide a general proof of this result for any C_ϕ such that

$C_\phi : B_r \to B_r$ $(1 < r < q)$ is bounded (see Proposition 3.6). The proof of Theorem 3.7 will now follow easily.

We next note that a theorem of Arazy, Fisher and Peetre (see Theorem F) can be used to characterize the boundedness of composition operators with domain the Bloch space and range inside a variety of spaces. For example in any Besov space, in $BMOA$, and in H^2 (see Proposition 3.8). Moreover we note that the integral condition of Shapiro and Taylor characterizing the Hilbert-Schmidt composition operators on the Dirichlet space (see [29]) also characterizes the bounded operators $C_\phi : \mathcal{B} \to \mathcal{D}$. We show that such operators are compact on $BMOA$. More general examples of compact C_ϕ on $BMOA$ are provided by integral conditions of this type. We prove the following proposition.

Proposition 3.9 *Let ϕ be a holomorphic self-map of U. Then,*

1. *If $1 < p < \infty$ then*

$$\int_U \frac{|\phi'(z)|^p (1-|z|^2)^{p-2}}{(1-|\phi(z)|^2)^p} dA(z) < \infty$$

if and only if $C_\phi : \mathcal{B} \to B_p$ is a compact operator (hence $C_\phi : \mathcal{B} \to BMOA$ is a compact operator as well).

2. *If*

$$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2 (1-\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z) = 0$$

then $C_\phi : \mathcal{B} \to BMOA$ is a compact operator.

Next we give a characterization of compact operators $C_\phi : X \to VMOA$, where X is a Möbius invariant subspace of the Bloch space. We prove the following theorem.

Theorem 3.11 *Let ϕ be a holomorphic self-map of U, and X a Möbius invariant*

Banach space. Then $C_\phi : X \to VMOA$ *is a compact operator if and only if*

$$\lim_{|q|\to 1} \sup_{\substack{||f||_X<1 \\ f\in X}} \int_U |f'(\phi(z))|^2 |\phi'(z)|^2 (1-|\alpha_q(z)|^2)\, dA(z) = 0.$$

Next we give an integral condition characterization of compact $C_\phi : \mathcal{B} \to VMOA$. The proof is similar to the one given by Arazy, Fisher, and Peetre in [2, Theorem 3] for characterizing Bloch Carleson measures. The main tools are Kintchine's inequality for gap series and Theorem 3.11. We prove the following theorem.

Theorem 3.13 *Let* ϕ *be a holomorphic self-map of* U*. Then the following are equivalent:*

1. $C_\phi : \mathcal{B} \to VMOA$ *is a compact operator.*

2. $$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2 (1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z) = 0\,.$$

Next we show that if ϕ is a boundedly valent holomorphic self-map of U such that $\phi(U)$ lies inside a polygon inscribed in the unit circle then the compactness of C_ϕ on Besov spaces, $BMOA$, and $VMOA$ is equivalent to the compactness of C_ϕ on the Bloch space. More precisely we prove the following theorem.

Theorem 3.15 *Let* ϕ *be a boundedly valent holomorphic self-map of* U *such that* $\phi(U)$ *lies inside a polygon inscribed in the unit circle. Then the following are equivalent:*

1. $C_\phi : \mathcal{B} \to VMOA$ *is a compact operator.*

2. $C_\phi : \mathcal{B} \to BMOA$ *is a compact operator.*

3. $C_\phi : BMOA \to BMOA$ *is a compact operator.*

4. $C_\phi : \mathcal{B} \to \mathcal{B}$ *is a compact operator.*

5. $C_\phi : \mathcal{B}_0 \to \mathcal{B}_0$ *is a compact operator.*

6. $C_\phi : VMOA \to VMOA$ *is a compact operator.*

The main tools of the proof are the following. First there is Madigan and Matheson's characterization of Bloch and little Bloch compactness. Next that boundedly valent holomorphic functions on the little Bloch space must belong to $VMOA$. Finally, we use Proposition 3.12 and Theorem 3.13.

In chapter 4 we give some final remarks and questions.

CHAPTER 1

Besov spaces, BMOA, and VMOA

Let U be the open unit disc in the complex plane and ∂U the unit circle. The one-to-one holomorphic functions that map U onto itself, called the *Möbius transformations*, and denoted by G, have the form

$$\lambda \alpha_p$$

where $\lambda \in \partial U$ and α_p is the basic conformal automorphism defined by

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p} z}$$

for $p \in U$. It is easy to check that the inverse of α_p under composition is α_p

$$\alpha_p \circ \alpha_p(z) = z$$

for $z \in U$. Also,

$$|\alpha_p'(z)| = \frac{1 - |p|^2}{|1 - \overline{p} z|^2}$$

and

$$1 - |\alpha_p(z)|^2 = \frac{(1 - |p|^2)(1 - |z|^2)}{|1 - \overline{p} z|^2} = (1 - |z|^2)|\alpha_p'(z)| \tag{1.1}$$

for $p, z \in U$.

The *Bloch space* $\mathcal{B}$ of U is the space of holomorphic functions f on U such that

$$||f||_{\mathcal{B}} = \sup_{z\in U}(1-|z|^2)|f'(z)| < \infty.$$

It is easy to see that $|f(0)|+||f||_{\mathcal{B}}$ defines a norm that makes the Bloch space a Banach space. Using (1.1) it is easy to see that $\mathcal{B}$ is invariant under Möbius transformations, that is, if $f \in \mathcal{B}$ then $f \circ \phi \in \mathcal{B}$, for all $\phi \in G$. In fact,

$$||f \circ \phi||_{\mathcal{B}} = ||f||_{\mathcal{B}}.$$

The polynomials are not dense in the Bloch space. The closure of the polynomials in the Bloch norm is called the *little Bloch space*, denoted by $\mathcal{B}_0$. In [34, page 84] is shown that

$$f \in \mathcal{B}_0 \text{ if and only if } \lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

A linear space X of holomorphic functions on U with a seminorm $||.||_X$ is *Möbius invariant* if

1. $X \subset \mathcal{B}$ and there exists a positive constant c such that for all $f \in X$,

$$||f||_{\mathcal{B}} \le c||f||_X.$$

2. For all $\phi \in G$ and all $f \in X$, $f \circ \phi \in X$ and

$$||f \circ \phi||_X = ||f||_X.$$

A *Möbius invariant* Banach space is a Möbius invariant linear space of holomorphic functions on U with a seminorm $||.||_X$, whose norm is $f \to ||f||_X$ or $f \to |f(0)|+||f||_X$.

For $1 < p < \infty$, the *Besov space* B_p is defined to be the space of holomorphic functions f on U such that

$$\begin{aligned} ||f||_{B_p}^p &= \int_U |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_U |f'(z)|^p (1-|z|^2)^p d\lambda(z) < \infty \end{aligned}$$

where $d\lambda(z)$ is the Möbius invariant measure on U, namely

$$d\lambda(z) = \frac{1}{(1-|z|^2)^2} dA(z).$$

It is easy to see that $|f(0)| + ||f||_{B_p}$ is a norm on B_p that makes it a Banach space.

It is easy to see that $\log(1-z) \in \mathcal{B}$. Moreover Holland and Walsh show in [13, Theorem 1] that if $1 < p < \infty$, and $\gamma < \frac{1}{q}$ (q is such that $\frac{1}{p} + \frac{1}{q} = 1$) then $\left(\log \frac{2}{1-z}\right)^\gamma \in B_p$. Other examples of functions in $\mathcal{B}, \mathcal{B}_0$, and B_p ($1 < p < \infty$) are provided by gap series. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n},$$

where (λ_n) is a sequence of integers satisfying

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \lambda > 1, \tag{1.2}$$

where λ is a constant and $n \in N$. Anderson, Clunie, and Pommerenke show in [1, Lemma 2.1] that $f \in \mathcal{B}$ if and only if $a_n = O(1)$, as $n \to \infty$, and that $f \in \mathcal{B}_0$ if and only if $a_n \to 0$, as $n \to \infty$. Moreover, a description of Besov spaces that Peller gives in [23, page 450] easily yields that $f \in B_p$ if and only if $\sum_{k=0}^{\infty} \lambda_k |a_k|^p < \infty$.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a holomorphic function on U. The *Hardy space* H^2 is the collection of functions f

holomorphic on U for which

$$||f||_{H^2}^2 \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

The *Dirichlet space* is the collection of functions f holomorphic on U for which

$$||f||_{\mathcal{D}}^2 \stackrel{\text{def.}}{=} \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

Both H^2 and $\mathcal{D}$ become Hilbert spaces with norms $||f||_{H^2}$ and $(|f(0)|^2 + ||f||_{\mathcal{D}}^2)^{\frac{1}{2}}$ respectively. It is easy to see, using polar coordinates, that $f \in \mathcal{D}$ if and only if

$$\int_U |f'(z)|^2 dA(z) < \infty.$$

Thus, the Besov-2 space is the Dirichlet space and $B_2 = \mathcal{D} \subset H^2$.

Let const. denote a positive and finite constant which may change from one occurence to the next but will not depend on the functions involved. Unlike the Hardy and Bergman spaces the Besov space with a smaller index lies inside the Besov space with a larger index.

Lemma 1.1 *For* $1 < p < q$, $B_p \subset B_q \subset \mathcal{B}$, *and for any* $f \in B_p$,

$$||f||_{\mathcal{B}} \leq \text{const.}||f||_{B_q} \leq \text{const.}||f||_{B_p}.$$

Proof. First, let us show that each Besov space lies inside the Bloch space. Fix $p > 1$, let $f \in B_p$; then,

$$\infty \quad > \quad \int_U |f'(z)|^p (1 - |z|)^2)^{p-2} dA(z)$$

$$\geq \int_R^1 \left\{ \int_0^{2\pi} |f'(re^{i\theta})|^p \frac{d\theta}{2\pi} \right\} (1-r^2)^{p-2} r dr$$

$$\geq \int_R^1 \left\{ \int_0^{2\pi} |f'(Re^{i\theta})|^p \frac{d\theta}{2\pi} \right\} (1-r^2)^{p-2} r dr$$

$$= \int_0^{2\pi} |f'(Re^{i\theta})|^p \frac{d\theta}{2\pi} \int_R^1 (1-r^2)^{p-2} r dr$$

$$= \frac{1}{2} \int_0^{2\pi} |f'(Re^{i\theta})|^p \frac{d\theta}{2\pi} \int_0^{1-R^2} r^{p-2} dr$$

$$\geq c \int_0^{2\pi} |f'(Re^{i\theta})|^p \frac{d\theta}{2\pi} (1-R)^{p-1} ,$$

where c is some positive constant, and $0 < R < 1$. Above we used the fact that the integral means of an analytic function f, $M_p(R, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta}|^p d\theta \right\}$ $(0 < p < \infty)$, are a non- decreasing function of R (Hardy's Convexity Theorem [11, page 9]). Thus,

$$M_p(R, f) \leq \text{const.} \frac{1}{(1-R)^{\frac{p-1}{p}}} = \frac{1}{(1-R)^{1-\frac{1}{p}}} .$$

Then by the Hardy-Littlewood theorem ([11, Theorem 5.9, page 84]), the infinity means of f',

$$M_\infty(R, f') = \max_{0 \leq \theta < 2\pi} |f'(Re^{i\theta})|,$$

can not grow faster than

$$\frac{1}{(1-R)^{1-\frac{1}{p}+\frac{1}{p}}} = \frac{1}{1-R}$$

that is

$$\sup_{\theta \in [0, 2\pi]} |f'(Re^{i\theta})| \leq c \frac{1}{1-R}$$

for some positive constant c. Now, it is easy to see that this implies that f belongs to the Bloch space, and

$$||f||_{B_p}^p \geq c||f||_{\mathcal{B}} \ .$$

Therefore, $B_p \subset \mathcal{B}$ for any $p > 1$.

Next, for the containment among Besov spaces, fix p and q such that $1 < p < q$ and let $f \in B_p$. Then,

$$\begin{aligned} ||f||_{B_q}^q &= \int_U |f'(z)|^q (1-|z|^2)^q d\lambda(z) \\ &= \int_U |f'(z)|^p (1-|z|^2)^p (|f'(z)|(1-|z|^2))^{p-q} d\lambda(z) \\ &\leq c||f||_{\mathcal{B}}^{q-p} ||f||_{B_p}^p < \infty. \end{aligned}$$

Thus, $B_p \subset B_q$. This finishes the proof of the lemma. □

Lemma 1.2 *For $1 < p < \infty$, B_p is a Möbius invariant Banach space.*

Proof. Let $f \in B_p$, $q \in U$. Then,

$$\begin{aligned} ||f \circ \alpha_q||_{B_p}^p &= \int_U |(f \circ \alpha_q)'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_U |f'(\alpha_q(z))|^p |\alpha_q'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_U |f'(w)^p |\alpha_q'(\alpha_q(w))|^p (1-|\alpha_q(w)|^2)^{p-2} |\alpha_q'(w)|^2 dA(w) \\ &= \int_U |f'(w)|^p \frac{1}{|\alpha_q'(w)|^p} (1-|w|^2)^{p-2} |\alpha_q'(w)|^{p-2} |\alpha_q'(w)|^2 dA(w) \\ &= \int_U |f'(w)|^p (1-|w|^2)^{p-2} dA(w) = ||f||_{B_p}^p. \end{aligned}$$

Above we made the change of variables $\alpha_q(z) = w$ and used basic properties of the Möbius transformations. This shows that B_p is invariant under Möbius transformations. Thus, by Lemma 1.1, B_p is a Möbius invariant Banach space. $\square$

A holomorphic function f on U belongs to $BMOA$, the holomorphic members of BMO, if

$$||f||_G = \sup_{q\in U} ||f \circ \alpha_q(z) - f(q)||_{H^2} < \infty. \tag{1.3}$$

Under the norm $|f(0)| + ||f||_G$ $BMOA$ becomes a complete normed linear space. This is not the traditional definition of $BMOA$, it is actually a corollary of the John-Nirenberg theorem [4, page 15]. By the Littlewood-Paley identities (see [34, page 167]) and the fact that $\log \frac{1}{|z|} \sim 1 - |z|^2$, for z away from the origin we see that a seminorm equivalent to the one defined in (1.3) is:

$$\begin{aligned} ||f||_*^2 &= \sup_{q\in U} \int_U |(f \circ \alpha_q)'(z)|^2 (1-|z|^2) dA(z) \\ &= \sup_{q\in U} \int_U |f'(\alpha_q(z))|^2 |\alpha_q'(z)|^2 (1-|z|^2) dA(z) \,. \end{aligned}$$

Thus after the change of variables $\alpha_q(z) = w$ we obtain

$$||f||_*^2 = \sup_{q\in U} \int_U |f'(w)|^2 (1 - |\alpha_q(w)|^2) dA(w). \tag{1.4}$$

Notation $S(h,\theta) = \{z \in U : |z - e^{i\theta}| < h\}$, where $\theta \in [0, 2\pi)$, $h \in (0,1)\}$.

Let A and B be two quantities that depend on a holomorphic function f on U. We say that A is *equivalent* to B, we write $A \sim B$, if

$$\text{const.}\, A \leq B \leq \text{const.}\, A \,.$$

The notion of $BMOA$ first arose in the context of mean oscillations of a function over cubes with edges parallel to the coordinate axes or equivalently over sets of the form $S(h,\theta)$ ([28, pages 36-39]). That is,

$$||f||_*^2 \sim \sup_{\substack{h\in(0,1)\\ \theta\in[0,2\pi)}} \frac{1}{h}\int_{S(h,\theta)} |f'(z)|^2(1-|z|^2)dA(z). \tag{1.5}$$

The function $\log(1-z) \in BMOA$. In fact if f is any holomorphic, univalent, and zero free function then $\log f \in BMOA$. (this result first appeared in [3] and [6]). Other examples of $BMOA$ functions include the following. If (a_n) is a bounded sequence then $\sum_{n=0}^{\infty} \frac{1}{n} a_n z^n \in BMOA$, and if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ then $\sum_{n=0}^{\infty} a_n z^{\lambda_n} \in BMOA$, where the sequence (λ_n) satisfies (1.2).

One of the many similarities between the Bloch space and $BMOA$ is that polynomials are not dense in either space. The closure of the polynomials in the $BMOA$ norm forms $VMOA$, the space of holomorphic functions with vanishing mean oscillation. The space $VMOA$ can be characterized as all those holomorphic functions f on U such that

$$\lim_{|q|\to 1} \int_U |f'(w)|^2(1-|\alpha_q(w)|^2)dA(w) = 0 \tag{1.6}$$

(the "little-oh" version of (1.4)). Moreover the "little-oh" version of (1.5) is equivalent to (1.6) ([28, pages 36-37, page 50]).

An easy way to see that $BMOA$ is a subspace of the Bloch space is the following:

$$|f'(0)| \le ||f||_{H^2}$$

for any f holomorphic on U; therefore,

$$\begin{aligned} |(f\circ\alpha_p - f(p))'(0)| &\le ||f\circ\alpha_p - f(p)||_{H^2} \\ &\le ||f||_G \end{aligned}$$

hence

$$|f'(\alpha_p(0)||\alpha_p'(0)| \leq \text{const.}||f||_*$$

that is

$$|f'(p)|(1-|p|^2) \leq \text{const.}||f||_*$$

thus,

$$||f||_{\mathcal{B}} \leq \text{const.}||f||_* \ .$$

Therefore, $BMOA \subset \mathcal{B}$.

Let H^∞ denote the space of bounded holomorphic functions on U.

Lemma 1.3 *The space $VMOA \cap H^\infty$ is closed under pointwise multiplication.*

Proof. Let $f, g \in VMOA \cap H^\infty$. Then,

$$\int_U |(fg)'(z)|^2(1-|\alpha_q(z)|^2)dA(z)$$

$$= \int_U |f'(z)|^2|g(z)|^2(1-|\alpha_q(z)|^2)dA(z) + \int_U |g'(z)|^2|f(z)|^2(1-|\alpha_q(z)|^2)dA(z)$$

$$\leq \text{const.}\left\{\int_U |f'(z)|^2(1-|\alpha_q(z)|^2)dA(z) + \int_U |g'(z)|^2(1-|\alpha_q(z)|^2)dA(z)\right\} .$$

The righthand side of the above equation tends to zero as $|q| \to 1$, since $f,\ g \in VMOA$. Hence, $fg \in VMOA$. □

Lemma 1.4 *For any $p > 1$, B_p is a subspace of VMOA.*

Proof. Fix $p > 2$; first we will show that $B_p \subset H^2$. Let $f \in B_p$. Then,

$$\int_U |f'(z)|^2(1-|z|^2)dA(z) \ = \ \int_U |f'(z)|^2(1-|z|^2)^2(1-|z|^2)d\lambda(z)$$

$$\leq \quad ||f||_{B_p}^2 (\int_U (1-|z|^2)^{\frac{p}{p-2}} d\lambda(z))^{\frac{p-2}{p}}$$

by Hölder's inequality. Since,

$$\int_U (1-|z|^2)^{\frac{p}{p-2}} d\lambda(z) = \int_U (1-|z|^2)^{\frac{4-p}{p-2}} dA(z) < \infty$$

for any $p > 2$,

$$\int_U |f'(z)|^2 (1-|z|^2) dA(z) < \infty.$$

Therefore, $B_p \subset H^2$, for any $p > 2$. Since $\mathcal{D} \subset H^2$, if $1 < p \leq 2$ then, $B_p \subseteq \mathcal{D} \subset H^2$. Therefore $B_p \subset H^2$, for any $p > 1$.

By the Möbius invariance of Besov spaces we obtain

$$||f \circ \alpha_q - f(q)||_{H^2}^2 \leq c||f \circ \alpha_q - f(q)||_{B_p}^p = c||f||_{B_p}^p$$

for some positive constant c and for any $q \in U$. Therefore,

$$||f||_*^2 \leq c||f||_{B_p}^p. \tag{1.7}$$

This shows that $B_p \subset BMOA$.

Next we show that polynomials are dense in B_p. This together with (1.7) then shows that $B_p \subset VMOA$. Let $f \in B_p$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $\sigma_n(f)$ the n-th Fejer mean of f, that is:

$$\sigma_n(f)(z) = \sum_{\lambda=0}^{n} (1 - \frac{|\lambda|}{n+1}) a_\lambda z^\lambda = \int_0^{2\pi} f(ze^{i\theta}) K_n(\theta) \frac{d\theta}{2\pi} \tag{1.8}$$

where $K_n(\theta)$ is Fejer's kernel,

$$K_n(\theta) = \sum_{\lambda=-n}^{n} (1 - \frac{|\lambda|}{n+1}) e^{-i\lambda\theta}. \tag{1.9}$$

We will show that $\sigma_n(f) \to f$ in B_p; Fubini's theorem yields,

$$\begin{aligned}
||\sigma_n(f) - f||_{B_p}^p &= \int_U |\sigma_n(f)'(z) - f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\leq \int_U \int_0^{2\pi} |e^{i\theta} f'(ze^{i\theta}) - f'(z)|^p K_n(\theta) \frac{d\theta}{2\pi} (1-|z|^2)^{p-2} dA(z) \\
&= \int_0^{2\pi} ||f(ze^{i\theta}) - f(z)||_{B_p}^p K_n(\theta) \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} g(e^{i\theta}) K_n(\theta) \frac{d\theta}{2\pi} \\
&= \sigma_n(g)(1), \qquad (1.10)
\end{aligned}$$

where $g(e^{i\theta}) = ||f(ze^{i\theta}) - f(z)||_{B_p}^p$. It is easy to see that g is a continuous function on ∂U. Therefore, by Theorem 2.11 in [14, page 15]

$$\lim_{n\to\infty} \max_{0\leq t\leq 2\pi} |\sigma_n(g)(e^{it}) - g(e^{it})| = 0.$$

Hence, $\sigma_n(g)(1) \to g(1) = 0$, as $n \to \infty$. Thus (1.10) yields,

$$\lim_{n\to\infty} ||\sigma_n(f) - f||_{B_p}^p = 0.$$

Therefore we obtain that $B_p \subset VMOA$. $\square$

We have shown that for $p < q$

$$B_p \subset B_q \subset VMOA \subset BMOA \subset \mathcal{B}.$$

Similarly to Lemma 1.2 we can show that $BMOA$ and $VMOA$ are also Möbius invariant Banach spaces. In fact, the reason for insisting that a Möbius invariant Banach space be a subspace of the Bloch space is that Rubel and Timoney proved in [26] that if a linear space of analytic functions on U with a seminorm $||.||_X$ is such that for all $f \in X$, $f \circ \phi \in X$ and $||f \circ \phi||_X = ||f||_X$, and it has a non-zero linear functional L that is *decent* (that is L extends to a continuous linear functional on the space of holomorphic functions on U) then, X has to be a subspace of the Bloch space and the inclusion map is continuous.

CHAPTER 2

Carleson measures and compact composition operators on Besov spaces and BMOA

If ϕ is a holomorphic self-map of U, then the composition operator C_ϕ

$$C_\phi f = f \circ \phi$$

maps holomorphic functions f to holomorphic functions.

Shapiro and Taylor show in [29], using the Riesz Factorization theorem and Vitali's convergence theorem that C_ϕ is compact on H^p, for some $0 < p < \infty$ if and only if C_ϕ is compact on H^2. Moreover, Shapiro solves the compactness problem for composition operators on H^p in [31] using the Nevanlinna counting function

$$N_\phi(w) = \sum_{\phi(z)=w} -\log|w|.$$

The following theorem is proved there;

Theorem A *Let ϕ be a holomorphic function on U. Then C_ϕ is a compact operator on H^2 if and only if*

$$\lim_{|w|\to 1} \frac{N_\phi(w)}{-\log|w|} = 0.$$

Madigan and Matheson characterize compact composition operators in the Bloch space in [22]. The following theorem is proved there;

Theorem B *Let ϕ be a holomorphic function on U. Then, C_ϕ is a compact operator on $\mathcal{B}$ if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0.$$

In this chapter we will use some Nevanlinna type functions to characterize the compact composition operators on Besov spaces $BMOA$, and $VMOA$.

Definition 2.1 *The counting function for the p-Besov space is*

$$N_p(w,\phi) = \sum_{\phi(z)=w} \left\{|\phi'(z)|(1-|z|^2)\right\}^{p-2}$$

for $w \in U, p > 1$.

Definition 2.2 *The counting functions for $BMOA$ are*

$$N(w,q,\phi) = \sum_{\phi(z)=w} (1-|\alpha_q(z)|^2)$$

for $w, q \in U$.

The above counting functions come up in the change of variables formula in the respective spaces as follows:

First, for $f \in B_p$ and $p > 1$

$$\begin{aligned} ||C_\phi f||_{B_p}^p &= \int_U |(f\circ\phi)'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_U |f'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^{p-2} dA(z). \end{aligned} \qquad (2.1)$$

By making a non-univalent change of variables as done in [32, page 186] we see that

$$||C_\phi f||_{B_p}^p = \int_U |f'(w)|^p N_p(w,\phi) dA(w). \qquad (2.2)$$

Similarly, for $BMOA$

$$\begin{aligned} ||C_\phi f||_*^2 &= \sup_{q\in U}\int_U |(f\circ\phi)'(z)|^2(1-|\alpha_q(z)|^2)dA(z) \\ &= \sup_{q\in U}\int_U |f'(\phi(z))|^2|\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z)\,. \end{aligned}$$

Thus,

$$||C_\phi f||_*^2 = \sup_{q\in U}\int_U |f'(w)|^2 N(w,q,\phi)dA(w). \tag{2.3}$$

Arazy, Fisher, and Peetre prove in [2, Theorem 12] that composition operators in $BMOA$ are bounded for any holomorphic self-map of U, and they are bounded on $VMOA$ if and only if the symbol belongs to $VMOA$. Next, we provide a proof similar to their proof.

Theorem C *Let ϕ be a holomorphic self-map of U. Then,*

1. *C_ϕ is a bounded operator on $BMOA$.*

2. *$C_\phi(VMOA)\subset VMOA$ if and only if $\phi\in VMOA$.*

Proof of *(1.)* Suppose that ϕ is a holomorphic self-map of U and $f\in BMOA$. If $\phi(0)=q\in U$ then $\phi=\alpha_q\circ\psi$ for some holomorphic self-map ψ of U such that $\psi(0)=0$. Then Littlewood's Subordination Principle (see [32, page 13]) yields

$$\begin{aligned} ||f\circ\phi - f(q)||_{H^2} &= ||f\circ\alpha_q\circ\psi - f(q)||_{H^2} \\ &\le ||f\circ\alpha_q - f(q)||_{H^2} \le ||f||_*\,. \end{aligned} \tag{2.4}$$

Thus replacing ϕ in (2.4) with $\phi\circ\alpha_q$ yields $||f\circ\phi\circ\alpha_q - f(q)||_{H^2}\le ||f||_*$ for all $q\in U$. Thus,

$$|f(\phi(0))| + ||f\circ\phi||_* \le \text{const.}(|f(0)| + ||f||_*),$$

for all $f \in BMOA$. This shows that C_ϕ is a bounded operator on $BMOA$.

Proof of *(2.)* First suppose that $C_\phi : VMOA \to VMOA$ is a bounded operator. Then since the identity function $f(z) = z$ belongs to $VMOA$, $f \circ \phi = \phi \in VMOA$. Conversely, suppose that $\phi \in VMOA$. Then, by Lemma 1.3, $\{\phi^n \in VMOA : n \in N\} \subset VMOA$. Therefore $\{p(\phi) : p \text{ polynomial}\} \subset VMOA$. Since polynomials are dense in VMOA part *(1)* above yields that $f \circ \phi \in VMOA$, for any $f \in VMOA$. This completes the proof of the theorem. □

Now consider the restriction of C_ϕ to B_p. Then C_ϕ is a bounded operator if and only if there is a positive constant c such that

$$||C_\phi f||_{B_p}^p \le c||f||_{B_p}^p$$

for all $f \in B_p$ or equivalently by (2.2)

$$\int_U |f'(w)|^p N_p(w, \phi) dA(w) \le c||f||_{B_p}^P$$

for all $f \in B_p$. This leads, as in [2], to the definition of Carleson type measures. Since we are interested in characterizing the compact composition operators we will also talk about vanishing Carleson measures. We would like to use the following operator theoretic wisdom;

If a "big-oh" condition characterizes the boundedness of an operator then the corresponding "little-oh" condition should characterize the compactness of the operator.

Definition 2.3 *Let μ be a positive measure on U and let $X = B_p$ $(1 < p < \infty)$, $BMOA$, or $\mathcal{B}$. Then μ is an (X, p)-Carleson measure if there is a constant $A > 0$ so* *that*

$$\int_U |f'(w)|^p d\mu(w) \leq A||f||_X^p,$$

for all $f \in X$.

In view of (2.2) and (2.3) above we see that C_ϕ is a bounded operator on B_p if and only if the measure $N_p(w, \phi)dA(w)$ is a (B_p, p)-Carleson measure, and C_ϕ is a bounded operator on $BMOA$ if and only if $N(w, q, \phi)dA(w)$ are uniformly $(BMOA, 2)$- Carleson measures.

Arazy, Fisher, and Peetre gave the following characterization of (B_p, p) Carleson measures in [2, Theorem 13] (the equivalence of *(1)* and *(2)* was given by Cima and Wogen in [7]).

Theorem D *For $1 < p < \infty$, the following are equivalent:*

1. *μ is a (B_p, p)-Carleson measure.*
2. *There exists a constant $A > 0$ such that*

   $$\mu(S(h, \theta)) \leq Ah^p$$

   for all $\theta \in [0, 2\pi)$, all $h \in (0, 1)$.
3. *There exists a constant $B > 0$ such that*

   $$\int_U |\alpha_q'(z)|^p d\mu(z) \leq B$$

   for all $q \in U$.

Hence Theorem D yields,

Theorem E *Let ϕ be a holomorphic function on U. Then C_ϕ is a bounded operator* *on* B_p *(*$1 < p < \infty$*) if and only if*

$$\sup_{q \in U} ||C_\phi \alpha_q||_{B_p} < \infty\,.$$

We prove a similar theorem for compact composition operators on Besov spaces.

Definition 2.4 *For* $1 < p < \infty$*,* μ *is called a vanishing* p*-Carleson measure if*

$$\lim_{h \to 0} \sup_{\theta \in [0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} = 0\,.$$

Note It is easy to see that if μ is a vanishing $p-$Carleson measure then it is a (B_p, p)-Carleson measure.

The proposition below characterizes vanishing p-Carleson measures. The proof is similar to the one for Carleson measures on H^2 ($p = 1$), as given by Garnett in [12] and by Chee in [5].

Proposition 2.5 *For* $1 < p < \infty$*, the following are equivalent:*

1. μ *is a vanishing* p*-Carleson measure.*

2. $\int_U |\alpha_q'(w)|^p d\mu(w) \to 0,$ *as* $|q| \to 1$*.*

Proof. First, suppose that *(2)* holds. Then, given an $\epsilon > 0$ there is a $\delta > 0$ such that for $1 - \delta < |q| < 1$

$$\int_U |\alpha_q'(w)|^p d\mu(w) < \epsilon.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be as above. Consider any $0 < h < \delta$, $\theta \in [0, 2\pi)$, let $q = (1-h)e^{i\theta}$ and $w \in S(h,\theta)$. Then,

$$\begin{aligned} |\alpha_q'(w)| &= \frac{1 - |q|^2}{|1 - \overline{q}w|^2} \\ &= \frac{1 - (1-h)^2}{|1 - (1-h)e^{-i\theta}w|^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{h(2-h)}{|e^{i\theta}-(1-h)w|^2} \\
&\geq \frac{h(2-h)}{(|e^{i\theta}-w|+|w-(1-h)w|)^2} \\
&\geq \frac{h(2-h)}{(h+h|w|)^2} \\
&= \frac{2-h}{h(1+|w|)^2} \\
&\geq \frac{1}{4h}\,.
\end{aligned}$$

Hence, $w \in S(h,\theta)$ implies that $|\alpha_q'(w)|^p \geq \frac{1}{4^p h^p}$. Then by our hypothesis,

$$\epsilon > \int_U |\alpha_q'(w)|^p d\mu \geq \int_{S(h,\theta)} |\alpha_q'(w)|^p d\mu \geq \frac{1}{4^p h^p}\mu(S(h,\theta)).$$

This proves *(1)*.

Conversely, suppose that *(1)* holds. Then, given an $\epsilon > 0$ there is a $\delta > 0$ such that for any $0 < h < \delta$ and any $\theta \in [0, 2\pi)$,

$$\mu(S(h,\theta)) < \epsilon h^p\,. \tag{2.5}$$

Fix $\epsilon > 0$, let δ be as above. Fix $h_0 < \delta$ such that (2.5) holds. Also, fix $q = |q|e^{i\theta} \in U$ with $|q| > 1 - \frac{h_0}{4}$. We will show that for q large,

$$\int_U |\alpha_q'(w)|^p d\mu(w) < \epsilon\,.$$

Let $E = \left\{w \in U : |e^{i\theta} - |q|w| \geq \frac{h_0}{4}\right\}$. Then for each $q \in U$,

$$\int_U |\alpha_q'(w)|^p d\mu(w) = \int_E |\alpha_q'(w)|^p d\mu(w) + \int_{E^c} |\alpha_q'(w)|^p d\mu(w)\,. \tag{2.6}$$

We will estimate each of the integrals above. First if $w \in E$,

$$|\alpha_q'(w)|^p = \left(\frac{1-|q|^2}{|e^{i\theta}-|q|w|^2}\right)^p \leq \left(4^2\,\frac{1-|q|^2}{h_o^2}\right)^p < \epsilon \tag{2.7}$$

for q large. Therefore (2.7) yields that for q large,

$$\int_E |\alpha_q'(w)|^p d\mu(w) < \epsilon\mu(E) \leq \mu(U)\epsilon < \text{const. } \epsilon\,. \tag{2.8}$$

Let $N = N(q)$ be the smallest positive integer such that

$$2^N(1-|q|) < h_0 \leq 2^{N+1}(1-|q|)\,. \tag{2.9}$$

We will show that

$$E^c \subset S(2^N(1-|q|),\theta) \subset S(h_0,\theta)\,. \tag{2.10}$$

Let $w \in E^c$. Then,

$$\begin{aligned} |w-e^{i\theta}| &= |w-e^{i\theta}+|q|w-|q|w| \\ &\leq |w-|q|w|+|e^{i\theta}-|q|w| \\ &< 1-|q|+\frac{h_0}{4} \\ &< 1-|q|+2^{N-1}(1-|q|) \\ &\leq 2^N(1-|q|)\,. \end{aligned}$$

This proves that $E^c \subset S(2^N(1-|q|),\theta)$. Next let $w \in S(2^N(1-|q|),\theta)$. Then, by (2.9)

$$|w-e^{i\theta}| \leq 2^N(1-|q|) < h_0\,.$$

Hence $S(2^N(1-|q|,\theta)) \subset S(h_0,\theta)$. Thus, (2.10) is proved.

Let $E_k = S(2^k(1-|q|),\theta)$, $k = 0, 1, ...N$. It is clear that

$$E_0 \subset E_1 \subset ... \subset E_N \subset S(h_0,\theta).$$

Then,

$$\begin{aligned}\int_{E^c} |\alpha_q'(w)|^p d\mu(w) &\leq \int_{S(2^N(1-|q|),\theta)} |\alpha_q'(w)|^p d\mu(w) \\ &= \int_{E_0} + \int_{E_1\setminus E_0} + ... + \int_{E_N\setminus E_{N-1}} |\alpha_q'(w)|^p d\mu(w)\,. \qquad (2.11)\end{aligned}$$

We will estimate each of the integrals above.

First, if $w \in E_0$ then $|w - e^{i\theta}| < 1 - |q|$ and

$$|\alpha_q'(w)| \leq \frac{1-|q|^2}{(1-|q|)^2} \leq \frac{2}{1-|q|}\,.$$

Since $1 - |q| < h_0 < \delta$, (2.5) yields

$$\begin{aligned}\int_{E_0} |\alpha_q'(w)|^p d\mu(w) &\leq \frac{2^p}{(1-|q|)^p}\mu(E_0) \\ &\leq \text{const.}\,\epsilon\,.\end{aligned}$$

Next if $w \in E_k \setminus E_{k-1}$ for some $k = 2, 3, ..., N$,

$$\begin{aligned}|\alpha_q'(w)| = \frac{1-|q|^2}{|e^{i\theta} - |q|w|^2} &\leq \frac{1-|q|^2}{(|w-e^{i\theta}| - |w|(1-|q|))^2} \\ &\leq \frac{\text{const.}}{1-|q|}\,\frac{1}{4^k}\,. \qquad (2.12)\end{aligned}$$

Hence, (2.5), (2.9), and (2.12) yield

$$\begin{aligned}
\int_{E_k\setminus E_{k-1}} |\alpha_q'(w)|^p d\mu(w) &\leq \frac{\text{const.}}{(1-|q|)^p}\,\frac{1}{4^{kp}}\,\mu(E_k) \\
&\leq \frac{\text{const.}}{(1-|q|)^p}\,\frac{1}{4^{kp}}\,\epsilon\,2^{kp}(1-|q|)^p \\
&= \text{const.}\,\frac{1}{2^{kp}}\,\epsilon\,. \qquad (2.13)
\end{aligned}$$

Therefore (2.8), (2.11), (2.12), and (2.13) imply that

$$\begin{aligned}
\int_U |\alpha_q'(w)|^p d\mu(w) &< \text{const.}\,\epsilon + \left(\sum_{k=0}^{N}\frac{1}{2^{kp}}\right)\text{const.}\,\epsilon \\
&< \text{const.}\,\epsilon
\end{aligned}$$

for q large. This proves *(2)* . □

Note In the proof above it was essential that

$$\sum_{k=0}^{\infty}\frac{1}{2^{kp}} < \infty,$$

since N depends on q.

The following is a corollary of the proof of Proposition 2.5. We will use it in the proof of Theorem 2.8.

Corollary 2.6 *Let $\{\mu_\lambda : \lambda \in I\}$ be a collection of positive measures. Then for $1 < p < \infty$ the following are equivalent:*

1.

$$\lim_{h\to 0} \sup_{\substack{\theta\in[0,2\pi)\\ \lambda\in I}} \frac{\mu_\lambda(S(h,\theta))}{h^p} = 0\,.$$

2.

$$\lim_{|q|\to 1} \sup_{\lambda\in I} \int_U |\alpha_q'(w)|^p d\mu_\lambda(w) = 0\,.$$

The following two theorems give a characterization of compact composition operators between Besov spaces, and from $\mathcal{D}$ to $BMOA$.

Theorem 2.7 *Let $1 < p \le q < \infty$. Then, the following are equivalent:*

1. *$C_\phi : B_p \to B_q$ is a compact operator.*

2. *$N_q(w,\phi)dA(w)$ is a vanishing q-Carleson measure.*

3. *$||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.*

Theorem 2.8 *The following are equivalent:*

1. *$C_\phi : \mathcal{D} \to BMOA$ is a compact operator.*

2. *$||C_\phi \alpha_\lambda||_* \to 0$, as $|\lambda| \to 1$.*

In the proof of the two theorems above we will need the following lemmas.

Lemma 2.9 *Let $X = B_p$ $(1 < p < \infty)$, $BMOA$, or $\mathcal{B}$. Then,*

1. *Every bounded sequence (f_n) in X is uniformly bounded on compact sets.*

2. *For any sequence (f_n) on X such that $||f_n||_X \to 0$, $f_n - f_n(0) \to 0$ uniformly on compact sets.*

Proof. In [34, page 82] is shown that a Bloch function can grow at most as fast as $\log \frac{1}{1-|z|}$, that is

$$|f_n(z) - f_n(0)| \;\le\; \text{const.}||f_n||_{\mathcal{B}} \, \log \frac{1}{1-|z|}$$

$$\leq \text{const.}||f_n||_X \ \log \frac{1}{1-|z|} \, .$$

Hence the result follows. □

Lemma 2.10 *Let X, Y be two Banach spaces of analytic functions on U. Suppose that*

1. *The point evaluation functionals on X are continuous.*

2. *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*

3. *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence (f_n) in X such that $f_n \to 0$ uniformly on compact sets, then the sequence (Tf_n) converges to zero in the norm of Y.

Proof. First, suppose that T is a compact operator and let (f_n) be a bounded sequence in X such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$. For the rest of this proof let $|.|_Y$ denote the norm of Y. If the conclusion is false then there exists an $\epsilon > 0$ and a subsequence $n_1 < n_2 < n_3 < \dots$ such that

$$|Tf_{n_j}|_Y \geq \epsilon, \text{ for all } j = 1, 2, 3, \dots \tag{2.14}$$

Since (f_n) is a bounded sequence and T a compact operator we can find a further subsequence $n_{j_1} < n_{j_2} < \dots$ and $f \in Y$ such that

$$|Tf_{n_{j_k}} - f|_Y \to 0 \, , \tag{2.15}$$

as $k \to \infty$. By *(1)* point evaluation functionals are continuous, therefore for any $z \in U$

$$|(Tf_{n_{j_k}} - f)(z)| \leq \text{const.}|Tf_{n_{j_k}} - f|_Y. \tag{2.16}$$

Hence (2.15) and (2.16) yield,

$$Tf_{n_{j_k}} - f \to 0 \tag{2.17}$$

uniformly on compact sets. Moreover, since $f_{n_{j_k}} \to 0$ uniformly on compact sets , *(3)* yields, $Tf_{n_{j_k}} \to 0$ uniformly on compact sets. Thus by (2.17) $f = 0$. Hence (2.15) yields $|Tf_{n_{j_k}}|_Y \to 0$ as $k \to \infty$, which contradicts (2.14). Therefore we must have $|Tf_n|_Y \to 0$, as $n \to \infty$.

Conversely, let (f_n) be a bounded sequence in X. We will show that the sequence (Tf_n) has a norm convergent subsequence. Without loss of generality (f_n) belongs to the unit ball of X. By *(2)* there is a subsequence $n_1 < n_2 < ...$ such that $f_{n_j} \to f$ uniformly on compact sets, for some $f \in X$. Hence, by our hypothesis, $|Tf_{n_j} - Tf|_Y \to 0$, as $j \to \infty$. This finishes the proof of the lemma. $\square$

Note ($\Rightarrow$) Only uses (*1*) and (*3*). ($\Leftarrow$) Only uses (*2*).

Lemma 2.11 *Let* $X, Y = B_p$ $(1 < p < \infty)$, $BMOA$, or $\mathcal{B}$. *Then* $C_\phi : X \to Y$ *is a compact operator if and only if for any bounded sequence* (f_n) *in* X *with* $f_n \to 0$ *uniformly on compact sets as* $n \to \infty$, $||C_\phi f_n||_Y \to 0$, *as* $n \to \infty$.

Proof. We will show that *(1), (2), (3)* of Lemma 2.10 hold for our spaces. By Lemma 2.9 it is easy to see that *(1)* and *(3)* hold. To show that *(2)* holds, let (f_n) be a sequence in the closed unit ball of X . Then by Lemma 2.9, (f_n) is uniformly bounded on compact sets. Therefore, by Montel's Theorem ([8, page 153]), there is a subsequence $n_1 < n_2 < ...$ such that $f_{n_k} \to g$ uniformly on compact sets, for some $g \in H(U)$. Thus we only need to show that $g \in X$.

(a) If $X = B_p$ $(1 < p < \infty)$,

$$\begin{aligned}
\int_U |g'(z)|^p(1-|z|^2)^{p-2}dA(z) &= \int_U \lim_{k\to\infty} |f'_{n_k}(z)|^p(1-|z|^2)^{p-2}dA(z) \\
&\leq \liminf_{k\to\infty} \int_U |f'_{n_k}(z)|^p(1-|z|^2)^{p-2} \\
&= \liminf_{k\to\infty} ||f_{n_k}||^p_{B_p} < \infty
\end{aligned}$$

by Fatou's Theorem and our hypothesis.

(b) If $X = BMOA$,

$$\begin{aligned}
\int_U |g'(z)|^2(1-|\alpha_q(z)|^2 dA(z) &= \int_U \lim_{k\to\infty} |f'_{n_k}(z)|^2(1-|\alpha_q(z)|^2 dA(z) \\
&\leq \liminf_{k\to\infty} \int_U |f'_{n_k}(z)|^2(1-|\alpha_q(z)|^2 dA(z) \\
&\leq \liminf_{k\to\infty} ||f_{n_k}||^2_* < \infty
\end{aligned}$$

by Fatou's Theorem and our hypothesis.

(c) If $X = \mathcal{B}$,

$$|g'(z)|(1-|z|^2) = \lim_{k\to\infty} |f'_{n_k}(z)|(1-|z|^2) \leq \lim_{k\to\infty} ||f_{n_k}||_{\mathcal{B}} < \infty$$

by our hypothesis. Therefore Lemma 2.10 yields that $C_\phi : X \to Y$ is a compact operator if and only if for any bounded sequence (f_n) in X with $f_n \to 0$ uniformly on compact sets as $n \to \infty$, $|f_n(\phi(0))| + ||C_\phi f_n||_Y \to 0$, as $n \to \infty$. Which is clearly equivalent to the statement of this lemma. This completes the proof of the lemma. $\square$

An immediate corollary of Lemma 2.11 is the following.

Corollary 2.12 *If ϕ is a holomorphic self-map of U such that $||\phi||_\infty < 1$ then C_ϕ is compact on every Besov space, and on $BMOA$.*

Proof. First, let us show that C_ϕ is compact on the Besov space B_p. Let (f_n) be a bounded sequence in B_p such that $f_n \to 0$ uniformly on compact subsets of U. Suppose that $\epsilon > 0$ is given. Since $\overline{\phi(U)}$ is a compact subset of U, there exists a positive integer N such that if $n \geq N$ then $|f_n'(\phi(z))|^p < \epsilon$, for all $z \in U$. Then by (2.1),

$$||C_\phi f_n||_{B_p}^p < \epsilon ||\phi||_{B_p}^p < \text{const. } \epsilon \, .$$

Thus, $||C_\phi f_n||_{B_p} \to 0$, as $n \to \infty$, and Lemma 2.11 yields that C_ϕ is a compact operator on B_p. The proof of the $BMOA$ compactness of C_ϕ is similar to the proof above. □

Now we are ready to prove Theorem 2.7 and 2.8. The technique is similar to the one given by Arazy, Fisher, and Peetre in [2, Theorem 13] and Luecking in [17], and [18].

Proof of Theorem 2.7. By (2.2),

$$||C_\phi \alpha_\lambda||_{B_q}^q = \int_U |\alpha_\lambda'(w)|^q N_\phi^q(w) dA(w) \, .$$

Thus Proposition 2.5 yields *(2)* $\Leftrightarrow$ *(3)*.

Next we show that *(1)* $\Rightarrow$ *(3)*. We assume that $C_\phi : B_p \to B_q$ is a compact operator. Note that $\{\alpha_\lambda : \lambda \in U\}$ is a bounded set in B_p since,

$$||\alpha_\lambda||_{B_p} = ||z \circ \alpha_\lambda||_{B_p} = ||z||_{B_p},$$

and the norm of α_λ in B_p is

$$|\alpha_\lambda(0)| + ||\alpha_\lambda||_{B_p} < 1 + ||z||_{B_p} < \infty.$$

Also $\alpha_\lambda - \lambda \to 0$, as $|\lambda| \to 1$, uniformly on compact sets since,

$$|\alpha_\lambda(z) - \lambda| = |z|\frac{1-|\lambda|^2}{|1-\bar{\lambda}z|}\,.$$

Hence, by Lemma 2.11, $||C_\phi(\alpha_\lambda - \lambda)||_{B_q} \to 0$, as $|\lambda| \to 1$. Therefore $||C_\phi \alpha_\lambda||_{B_q} \to 0$, as $|\lambda| \to 1$.

Finally, let us show that *(2)* $\Rightarrow$ *(1)*. Let (f_n) be a bounded sequence in B_p, that converges to 0, uniformly on compact sets. Then the mean value property for the holomorphic function f_n' yields,

$$f_n'(w) = \frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}} f_n'(z)dA(z). \tag{2.18}$$

Therefore by Jensen's inequality ([27, Theorem 3.3, page 62] and (2.18)),

$$|f_n'(w)|^q \leq \frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}} |f_n'(z)|^q dA(z). \tag{2.19}$$

Then by (2.19) and Fubini's Theorem ([27, Theorem 8.8, page 164]),

$$||C_\phi f_n||_{B_q}^q = \int_U |f_n'(w)|^q N_q(w,\phi)dA(w)$$

$$\leq \int_U \frac{4}{\pi(1-|w|)^2}\left(\int_{|w-z|<\frac{1-|w|}{2}} |f_n'(z)|^q dA(z)\right) N_q(w,\phi)dA(w)$$

$$= \frac{4}{\pi}\int_U |f_n'(z)|^q \left(\int_U \frac{1}{(1-|w|)^2}\chi_{\{z:|w-z|<\frac{1-|w|}{2}\}}(z)N_q(w,\phi)dA(w)\right) dA(z) \tag{2.20}$$

Note that if $|w-z|<\frac{1-|w|}{2}$ then $w\in S(2(1-|z|),\theta)$, where $z=|z|e^{i\theta}$, since

$$|w-e^{i\theta}|\le|z-w|+|e^{i\theta}-z|<\frac{1-|w|}{2}+|\frac{z}{|z|}-z|<2(1-|z|).$$

Moreover, if $|w-z|<\frac{1-|w|}{2}$ then $\frac{1}{(1-|w|)^2}\le \text{const.}\frac{1}{(1-|z|)^2}$. Therefore (2.20) yields,

$$\begin{aligned}||C_\phi f_n||_{B_q}^q &\le \text{const.}\int_U \frac{|f_n'(z)|^q}{(1-|z|)^2}\left(\int_{S(2(1-|z|),\theta)} N_q(w,\phi)dA(w)\right)dA(z)\\ &= \text{const.}\left(\int_{|z|>1-\frac{\delta}{2}}+\int_{|z|\le 1-\frac{\delta}{2}}\frac{|f_n'(z)|^q}{(1-|z|)^2}\left(\int_{S(2(1-|z|,\theta)} N_q(w,\phi)dA(w)\right)dA(z)\right)\\ &= \text{const.}(I+II)\,, \qquad (2.21)\end{aligned}$$

for any $0<\delta<1$.

Fix $\epsilon>0$ and let $\delta>0$ be such that for any $\theta\in[0,2\pi]$ and any $h<\delta$

$$\int_{S(h,\theta)} N_q(w,\phi)dA(w)<\epsilon h^q. \qquad (2.22)$$

By (2.21) and (2.22)

$$\begin{aligned}I &\le 2^q\epsilon\int_{|z|>1-\frac{\delta}{2}}\frac{|f_n'(z)|^q}{(1-|z|^2)^2}(1-|z|^2)^q dA(z)\\ &\le \text{const.}\epsilon||f_n||_{B_q}^q<\text{const.}\ \epsilon. \qquad (2.23)\end{aligned}$$

By (2.21),

$$II\ \le\ \text{const.}\int_{|z|\le 1-\frac{\delta}{2}}|f_n'(z)|^q\left(\int_U N_q(w,\phi)dA(w)\right)dA(z)$$

$$= \int_{|z|\leq 1-\frac{\delta}{2}} |f_n'(z)|^q ||\phi||_{B_q}^q dA(z) < \text{const. } \epsilon \tag{2.24}$$

for n large enough, since $f_n' \to 0$ uniformly on compact sets. Combining (2.21), (2.23) and (2.24) we obtain that $||C_\phi f_n||_{B_q} <$ const. ϵ for n large enough. Therefore $||C_\phi f_n||_{B_q} \to 0$, as $n \to \infty$ and Lemma 2.11 yields, $C_\phi : B_p \to B_q$ is a compact operator. This finishes the proof of Theorem 2.7. $\square$

Proof of Theorem 2.8. *(1) ⇒ (2).* Since α_λ is a bounded set in $\mathcal{D}$ and $\alpha_\lambda - \lambda \to 0$ uniformly on compact sets, as $|\lambda| \to 1$, Lemma 2.11 yields $||C_\phi \alpha_\lambda||_* \to 0$, as $|\lambda| \to 1$.

(2) ⇒ (1). The proof is similar to the proof of Theorem 2.7. We will use Lemma 2.11. Let (f_n) be a bounded sequence in $\mathcal{D}$ such that $f_n \to 0$ uniformly on compact sets. Our hypothesis is that $||C_\phi \alpha_\lambda||_* \to 0$, as $|\lambda| \to 1$. That is

$$\sup_{q\in U} \int_U |\alpha_\lambda'(w)|^2 N(w,q,\phi) dA(w) \to 0$$

as $|\lambda| \to 1$. Hence, Corollary 2.6 yields

$$\lim_{h\to 0} \sup_{\substack{q\in U \\ \theta\in[0,2\pi)}} \frac{1}{h^2} \int_{S(h,\theta)} N(w,q,\phi) dA(w) = 0.$$

Fix an $\epsilon > 0$ and let $\delta > 0$ be such that for any $\theta \in [0, 2\pi)$ and any $q \in U$, if $h < \delta$ then

$$\int_{S(h,\theta)} N(w,q,\phi) dA(w) < \epsilon h^2. \tag{2.25}$$

Fix $q \in U$. Then by (2.19),

$$\int_U |f_n'(w)|^2 N(w,q,\phi) dA(w)$$

$$\leq \int_U \frac{4}{\pi(1-|w|)^2} \left(\int_{|w-z|<\frac{1-|w|}{2}} |f_n'(z)|^2 dA(z) \right) N(w,q,\phi) dA(w)$$

$$\leq \text{const.} \int_U \frac{|f_n'(z)|^2}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} N(w,q,\phi) dA(w) \right) dA(z)\,. \tag{2.26}$$

The proof of (2.26) is the same as the proof of (2.21) in Theorem 2.7. Next split the integral in (2.26) into two pieces, one over the set $\{z \in U : |z| > 1 - \frac{\delta}{2}\}$ and the other over the complementary set . Then ,

$$\int_{|z|>1-\frac{\delta}{2}} \frac{|f_n'(z)|^2}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} N(w,q,\phi) dA(w) \right) dA(z)$$

$$< \epsilon \int_{|z|>1-\frac{\delta}{2}} \frac{|f_n'(z)|^2}{(1-|z|)^2} 4(1-|z|^2)^2 dA(z)$$

$$< \text{const.}\ \epsilon ||f_n||_{\mathcal{D}}^2 < \text{const.}\ \epsilon\,, \tag{2.27}$$

and

$$\int_{|z|\leq 1-\frac{\delta}{2}} \frac{|f_n'(z)|^2}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} N(w,q,\phi) dA(w) \right) dA(z)$$

$$\leq \text{const.} \left(\sup_{q\in U} \int_U N(w,q,\phi) dA(w) \right) \int_{|z|\leq 1-\frac{\delta}{2}} |f_n'(z)|^2 dA(z)$$

$$\leq \text{const.}\ \epsilon, \tag{2.28}$$

for n large enough since $\phi \in BMOA$ and $f_n' \to 0$ uniformly on $\left\{z \in U : |z| \leq 1 - \frac{\delta}{2}\right\}$. Therefore (2.26), (2.27), and (2.28) yield that

$$\sup_{q\in U} \int_U |f_n'(w)|^2 N(w,q,\phi) dA(w) < \text{const.}\ \epsilon$$

for n large enough. Thus $||C_\phi f_n||_* \to 0$, as $n \to \infty$. Hence by Lemma 2.11, *(1)* holds.

This finishes the proof of the theorem. □

Note It is easy to see that Theorem 2.8 yields that $C_\phi : B_p \to BMOA$ is a compact operator if and only if $||C_\phi \alpha_\lambda||_* \to 0$ as $|\lambda| \to 1$, for $1 < p \leq 2$. Moreover in chapter three we will show that if C_ϕ is bounded on some Besov space then this is valid for any $p > 1$.

The following is a corollary of the proof of the Theorem 2.8.

Corollary 2.13 *If* $\sup_{q\in U} \int_U |\alpha'_\lambda(w)|^3 N(w,q,\phi) dA(w) \to 0,$ *as* $|\lambda| \to 1,$ *then* $C_\phi : BMOA \to BMOA$ *is a compact operator.*

Note Similarly to the proof of the above theorems we can easily see that, the above sufficient condition for $BMOA$ compactness is equivalent to $C_\phi : H^2 \to BMOA$ being a compact operator.

CHAPTER 3

Besov space, BMOA, and VMOA compactness of $\mathbf{C}_\phi$ versus Bloch compactness of $\mathbf{C}_\phi$

In this chapter we give conditions that relate the compact composition operators on Besov spaces, $BMOA$, and $VMOA$ with those on the Bloch space, and the little Bloch space. Recall the characterization of compact composition operators on the Bloch space that Madigan and Matheson give in [22, Theorem 2].

Theorem B *Let ϕ be a holomorphic self-map of U. Then, C_ϕ is a compact operator on $\mathcal{B}$ if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0.$$

Next we give another characterization of compact composition operators on the Bloch space.

Theorem 3.1 *Let ϕ be a holomorphic self-map of U. Let $X = B_p$ $(1 < p < \infty)$, $BMOA$, or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is a compact operator if and only if*

$$\lim_{|\lambda|\to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0.$$

Proof. First, suppose that $C_\phi : X \to \mathcal{B}$ is a compact operator. Then $\{\alpha_\lambda : \lambda \in U\}$ is a bounded set in X, and $\alpha_\lambda - \lambda \to 0$ uniformly on compact sets as $|\lambda| \to 1$. Thus by Lemma 2.11

$$\lim_{|\lambda|\to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0.$$

Conversely, suppose that $\lim ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0$, as $|\lambda| \to 1$. Let (f_n) be a bounded sequence in X such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$. We will show that

$$\lim_{n\to\infty} ||C_\phi f_n||_{\mathcal{B}} = 0.$$

Let $\epsilon > 0$ be given and fix $0 < \delta < 1$ such that if $|\lambda| > \delta$ then $||C_\phi \alpha_\lambda||_{\mathcal{B}} < \epsilon$. Hence for any $z_0 \in U$ such that $|\phi(z_0)| > \delta$, $||C_\phi \alpha_{\phi(z_0)}||_{\mathcal{B}} < \epsilon$. In particular,

$$|\alpha'_{\phi(z_0)}(\phi(z_0))|\,|\phi'(z_0)|(1-|z_0|^2) < \epsilon$$

that is,

$$\frac{|\phi'(z_0)|}{1-|\phi(z_0)|^2}(1-|z_0|^2) < \epsilon\,. \tag{3.1}$$

Then (3.1) yields that for any $n \in N$ and $z_0 \in U$ such that $|\phi(z_0)| > \delta$,

$$\begin{aligned} |(f_n\circ\phi)'(z_0)|(1-|z_0|^2) &= |f'_n(\phi(z_0))|\,|\phi'(z_0)|(1-|z_0|^2) \\ &< |f'_n(\phi(z_0))|(1-|\phi(z_0)|^2)\,\epsilon \\ &\le ||f_n||_{\mathcal{B}}\,\epsilon \\ &\le ||f_n||_X\,\epsilon < \text{const.}\epsilon\,. \end{aligned} \tag{3.2}$$

Since the set $A = \{w : |w| \le \delta\}$ is a compact subset of U and $f'_n \to 0$ uniformly on compact sets,

$$\sup_{w\in A} |f'_n(w)| \to 0, \text{ as } n \to \infty\,.$$

Therefore we may choose N large such that $|f'_n(\phi(z))| < \epsilon$, for any $n \ge N$ and any

$z \in U$ such that $|\phi(z)| \le \delta$. Then, for all such z,

$$\begin{aligned} |(f_n \circ \phi)'(z)|(1-|z|^2) &= |f_n'(\phi(z))|\,|\phi'(z)|(1-|z|^2) \\ &< \epsilon|\phi'(z)|(1-|z|^2) \\ &< ||\phi||_{\mathcal{B}}\,\epsilon\,, \end{aligned} \tag{3.3}$$

where $n \ge N$. Thus, (3.2) and (3.3) yield

$$||f_n \circ \phi||_{\mathcal{B}} < \text{const.}\,\epsilon, \text{ for } n \ge N. \tag{3.4}$$

Thus (3.4) yields that $||C_\phi f_n||_{\mathcal{B}} \to 0$ as $n \to \infty$. Hence by Lemma 2.11 $C_\phi : X \to \mathcal{B}$ is a compact operator. $\square$

Notes (a) It is easy to see that the proof of Theorem 3.1 yields that

$$\lim_{|\lambda| \to 1} ||C_\phi \alpha_\lambda||_{\mathcal{B}} = 0$$

if and only if

$$\lim_{|\phi(z)| \to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0\,.$$

Therefore we obtain another proof of Theorem B.

(b) The above theorem is valid for any Banach subspace X of the Bloch space such that the point evaluation functionals on X are continuous and the closed unit ball of X is compact in the topology of uniform convergence on compact sets.

An immediate consequence of Theorem 3.1 along with Lemma 2.11 and Theorems 2.7 and 2.8 is the following proposition.

Proposition 3.2 *Let $1 < p \le q \le \infty$. Then:*

1. *If $C_\phi : B_p \to B_q$ is a compact operator then so is $C_\phi : \mathcal{B} \to \mathcal{B}$.*

2. For $1 < p \leq 2$,

if $C_\phi : B_p \to BMOA$ *is a compact operator then so is* $C_\phi : \mathcal{B} \to \mathcal{B}$.

3. If $C_\phi : BMOA \to BMOA$ *is a compact operator then so is* $C_\phi : \mathcal{B} \to \mathcal{B}$.

The following proposition gives a sufficient condition for a composition operator to be compact on a Besov space.

Proposition 3.3 *Let* $1 < p \leq q < \infty$. *If*

$$\lim_{|w|\to 1} \frac{N_q(w,\phi)}{(1-|w|^2)^{q-2}} = 0$$

then $C_\phi : B_p \to B_q$ *is a compact operator.*

Proof. Let (f_n) be a bounded sequence in B_p such that $f_n \to 0$ uniformly on compact sets as $n \to \infty$. Let $\epsilon > 0$ be given and fix $\delta > 0$ such that if $1-\delta < |w| < 1$ then

$$N_q(w,\phi) < \epsilon(1-|w|^2)^{q-2}. \tag{3.5}$$

By (2.2)

$$\begin{aligned} ||C_\phi f_n||_{B_q}^q &= \int_U |f_n'(w)|^q N_q(w,\phi) dA(w) \\ &= \int_{1-\delta<|w|<1} + \int_{|w|\leq 1-\delta} |f_n'(w)|^q N_q(w,\phi) dA(w) \\ &= I + II \end{aligned} \tag{3.6}$$

By (3.5),

$$I < \epsilon \int_{1-\delta<|w|<1} |f_n'(w)|^q (1-|w|^2)^{q-2} dA(w)$$

$$< \epsilon ||f_n||_{B_q}^q < \epsilon \text{ const. } \ (f_n \text{ is bounded in } B_p) \tag{3.7}$$

Since $|f_n'|^q \to 0$ uniformly on $\{w \in U : |w| \leq 1 - \delta\}$, we can find a positive integer N such that

$$II \leq \epsilon \int_{|w|<1-\delta} N_q(w, \phi) dA(w) < \epsilon \, \text{const.}\,, \tag{3.8}$$

for $n \geq N$, since

$$\int_{|w|<1-\delta} N_q(w, \phi) dA(w) \leq ||\phi||_{B_q} < \infty.$$

By (3.6), (3.7), and (3.8) $||C_\phi f_n||_{B_q} < \epsilon \, \text{const.}$ for $n \geq N$. Therefore, $||C_\phi f_n||_{B_q} \to 0$ as $n \to \infty$. Hence Lemma 2.11 yields $C_\phi : B_p \to B_q$ is a compact operator. □

Composition operators on Besov spaces are not bounded for all holomorphic self-maps of U. But if the Besov space contains the Dirichlet space and the symbol is boundedly valent then the induced composition operator is bounded.

Lemma 3.4 *Let ϕ be a boundedly valent holomorphic self-map of U, $2 \leq q < \infty$, and $1 < p \leq q$. Then $C_\phi : B_p \to B_q$ is a bounded operator.*

Proof. Let $f \in B_p$ $(1 < p < \infty)$. Applying the Schwarz Lemma ([27, page 254]) to the function $\alpha_z \circ \phi \circ \alpha_{\phi(z)}$ yields

$$|\phi'(z)|(1 - |z|^2) \leq 1 - |\phi(z)|^2,$$

for any $z \in U$. Hence by (2.2),

$$\begin{aligned} ||C_\phi f||_{B_q}^q &= \int_U |f'(w)|^q \sum_{\phi(z)=w} \left(|\phi'(z)|(1-|z|^2)\right)^{q-2} dA(w) \\ &\leq \text{const.} \int_U |f'(w)|^q \sum_{\phi(z)=w} \left(1 - |\phi(z)|^2\right)^{q-2} dA(w)\,. \end{aligned}$$

Therefore,

$$||C_\phi f||_{B_q}^q \leq \text{const.} \int_U |f'(w)|^q (1-|w|^2)^{q-2} dA(w) \leq \text{const.} ||f||_{B_p}^p$$

for any holomorphic function f on U. Thus, $C_\phi : B_p \to B_q$ is a bounded operator. □

The following theorem and proposition give conditions under which compactness in the Bloch space is equivalent to compactness from a Besov space to some larger Besov space.

Theorem 3.5 *Let ϕ be a univalent holomorphic self-map of U. Then, for $q > 2$, $C_\phi : B_q \to B_q$ is a compact operator if and only if $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

Proof. First, suppose that C_ϕ is a compact operator on the Bloch space. The sufficient condition of Besov space compactness in Proposition 3.3 for a univalent function is

$$\lim_{|w|\to 1} \left\{ \frac{|\phi'(\phi^{-1}(w))|(1-|\phi^{-1}(w)|^2}{1-|w|^2} \right\}^{q-2} = 0$$

or equivalently,

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0.$$

Which is a compactness condition for the composition operator on the Bloch space (Theorem B). Hence, by our assumption, $C_\phi : B_q \to B_q$ is a compact operator.

The converse follows from Proposition 3.2. This finishes the proof of the theorem. □

Note Theorem 3.5 is not valid when $q = 2$. There exists a univalent holomorphic self-map of U such that C_ϕ is compact on the Bloch space but not on the Dirichlet space. To describe such an example we will need some preliminaries. First, the Koebe Distortion Theorem (see [32, page 156]) which asserts that if ϕ is a univalent function on U then for any $z \in U$

$$\delta_{\phi(U)}(\phi(z)) \sim |\phi'(z)|(1-|z|^2),$$

where $\delta_{\phi(U)}(\phi(z))$ is the Euclidean distance from $\phi(z)$ to $\partial\phi(U)$. Thus the Madigan and Matheson condition of Bloch compactness for a univalent ϕ is equivalent to

$$\lim_{|\phi(z)|\to 1} \frac{\delta_{\phi(U)}(\phi(z))}{1-|\phi(z)|^2} = 0\,. \tag{3.9}$$

Let $D(0,\alpha)$ denote the disc centered at 0 of radius α. A *nontangential approach region* Ω_α $(0<\alpha<1)$ in U, with vertex $\zeta \in \partial U$ is the convex hull of $D(0,\alpha) \cup \{\zeta\}$ minus the point ζ.

If ψ is a univalent holomorphic self-map of U such that $\psi(U) = \Omega_\alpha$ $(0<\alpha<1)$ then $\inf_{z\in U} \frac{\delta_{\psi(U)}(\psi(z))}{1-|\psi(z)|^2} > 0$. Thus by (3.9) C_ψ is not compact on $\mathcal{B}$. But if we delete certain circular arcs from Ω_α then for the Riemann map ϕ from U onto the induced domain G, C_ϕ is compact on $\mathcal{B}$. Let $L_n = \{z \in \Omega_\alpha : |z-1| \le \frac{1}{2^n}\}$ $(n \ge 1)$. Then $L_1 \supset L_2 \supset L_3 \supset \ldots$. Remove from $L_n \setminus L_{n+1}$ $(n \ge 1)$ arcs centered at 1, with one end point at $\partial\Omega_\alpha$, in such a way so that the succesive radii are less than $\frac{1}{3^n}$ apart, and the distance of each arc to $\partial\Omega_\alpha$ is less than $\frac{1}{3^n}$. Then the distance from each $z \in L_n \setminus L_{n+1}$ to the boundary of the induced subdomain, G_n, of $L_n \setminus L_{n+1}$ is less than $\frac{1}{3^n}$. Let $G = \cup_{n\ge1} G_n$. Then, as $|z| \to 1$, $\delta_G(z) = o(1-|z|)$. Therefore by (3.9) C_ϕ is compact on $\mathcal{B}$. Moreover C_ϕ is not compact on the Dirichlet space. This follows from Theorem 2.7.

The theorem above is a special case of the following proposition. We show that if C_ϕ is bounded on some Besov space then the compactness of C_ϕ on larger Besov spaces is equivalent to the compactness of C_ϕ on the Bloch space. This result is similar to the compactness of C_ϕ on weighted Dirichlet spaces $\mathcal{D}_\alpha$ $(\alpha > -1)$. These are spaces of holomorphic functions f on U such that $|f(0)|^2 + \int_U |f'(z)|^2(1-|z|^2)^\alpha dA(z) < \infty$. MacCluer and Shapiro show in [19, Main Theorem, page 893] that if C_ϕ is bounded on some weighted Dirichlet space $\mathcal{D}_\alpha$ then the compactness of C_ϕ on larger weighted Dirichlet spaces is equivalent to ϕ having no angular derivative at each point of ∂U.

Proposition 3.6 *Let $1 < r < q$, $1 < p \le q$. Suppose that $C_\phi : B_r \to B_r$ is a bounded operator. Then, $C_\phi : B_p \to B_q$ is a compact operator if and only if $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

Proof . First, suppose that C_ϕ is a compact operator on the Bloch space. For any $\lambda \in U$,

$$\begin{aligned} ||C_\phi \alpha_\lambda||_{B_q}^q &= \int_U |\alpha_\lambda'(\phi(z))|^q |\phi'(z)|^q (1-|z|^2)^{q-2} dA(z) \qquad (3.10) \\ &= \int_U |\alpha_\lambda'(\phi(z))|^r |\phi'(z)|^r (1-|z|^2)^{r-2} \left(|\alpha_\lambda'(\phi(z))|\,|\phi'(z)|(1-|z|^2)\right)^{q-r} dA(z) \\ &\le ||C_\phi \alpha_\lambda||_{\mathcal{B}}^{q-r} ||C_\phi \alpha_\lambda||_{B_r}^r \\ &\le \text{const.}||C_\phi \alpha_\lambda||_{\mathcal{B}}^{q-r} \text{ (by Theorem } E \text{ and since } C_\phi : B_r \to B_r \text{ is bounded)}. \end{aligned}$$

Therefore (3.10) and Theorem 3.1 yield that $||C_\phi \alpha_\lambda||_{B_q} \to 0$ as $|\lambda| \to 1$. Thus by Theorem 2.7, $C_\phi : B_p \to B_q$ is a compact operator. The converse follows from Proposition 3.2. This finishes the proof of the proposition. □

The following theorem summarizes the above. If a composition operator is bounded on some Besov space then the compactness of the operator on larger Besov spaces, and from any Besov space to $BMOA$, is equivalent to the Bloch compactness of the operator.

Theorem 3.7 *Let* $1 < r < q$, $1 < p \leq q$, *suppose that* $C_\phi : B_r \to B_r$ *is a bounded operator. Then the following are equivalent:*

1. $C_\phi : \mathcal{B} \to \mathcal{B}$ *is a compact operator.*

2. $C_\phi : B_p \to B_q$ *is a compact operator.*

3. $C_\phi : \mathcal{D} \to BMOA$ *is a compact operator.*

4. $C_\phi : B_p \to BMOA$ *is a compact operator.*

Proof. The previous proposition yields (*1*) $\Leftrightarrow$ (*2*). Proposition 3.2 yields (*3*) $\Rightarrow$ (*1*). Theorems 2.7 and 2.8 yield (*2*) $\Rightarrow$ (*3*). Theorem 2.8 yields (*3*) $\Leftrightarrow$ (*4*), if $1 < p < 2$. If $p > 2$ then (*4*) $\Rightarrow$ (*3*) is trivial, since the inclusion map, $i : B_p \to \mathcal{D}$, is bounded. Moreover (*2*) $\Rightarrow$ (*4*) follows as well (when $p > 2$) since the inclusion map, $i : B_q \to BMOA$, is bounded. We have shown (*3*) $\Rightarrow$ (*1*) $\Leftrightarrow$ (*2*) $\Rightarrow$ (*3*) $\Leftrightarrow$ (*4*). This completes the proof of the theorem. $\square$

Arazy, Fisher, and Peetre prove the following theorem in [2, Theorem 16].

Theorem F *Let* μ *be a positive measure on* U, $0 < p < \infty$. *Then,*

$$\int_U \frac{d\mu(z)}{(1 - |z|^2)^p} < \infty$$

if and only if there is a positive constant c *such that*

$$\int_U |f'(z)|^p d\mu(z) \leq c||f||_{\mathcal{B}},$$

for all $f \in \mathcal{B}$.

Note The proof of Theorem F can be used to show that a similar result holds for a collection of positive measures $\{\mu_q : q \in U\}$. That is, if $0 < p < \infty$, then

$$\sup_{q\in U} \int_U \frac{d\mu_q}{(1-|z|^2)^p} < \infty$$

if and only if

$$\sup_{q\in U} \int_U |f'(z)|^p d\mu_q \leq c||f||_{\mathcal{B}}$$

for all $f \in \mathcal{B}$.

These results, along with a non-univalent change of variables, yield the following characterizations of bounded composition operators from the Bloch space to B_p ($1 < p < \infty$), $BMOA$, and H^2.

Proposition 3.8 *Let ϕ be a holomorphic self-map of U.*

1. *$C_\phi : \mathcal{B} \to \mathcal{D}$ is a bounded operator if and only if*

$$\int_U \frac{\eta(\phi; w)}{(1-|w|^2)^2} dA(w) = \int_U \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2} dA(z) < \infty \, ,$$

where $\eta(\phi; w)$ denotes the number of times ϕ takes the value w. If w is not in $\phi(U)$ then let $\eta(\phi; w) = 0$.

2. *$C_\phi : \mathcal{B} \to B_p$ ($1 < p < \infty$) is a bounded operator if and only if*

$$\int_U \frac{N_p(w, \phi)}{(1-|w|^2)^p} dA(w) = \int_U \frac{|\phi'(z)|^p (1-|z|^2)^{p-2}}{(1-|\phi(z)|^2)^p} dA(z) < \infty.$$

3. *$C_\phi : \mathcal{B} \to BMOA$ is a bounded operator if and only if*

$$\sup_{q\in U} \int_U \frac{N(w, q, \phi)}{(1-|w|^2)^2} dA(w) = \sup_{q\in U} \int_U \frac{|\phi'(z)|^2 (1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(w) < \infty.$$

4. *$C_\phi : \mathcal{B} \to H^2$ is a bounded operator if and only if*

$$\int_U \frac{N(w,0,\phi)}{(1-|w|^2)^2} dA(w) = \int_U \frac{|\phi'(z)|^2(1-|z|^2)}{(1-|\phi(z)|^2)^2} dA(w) < \infty.$$

Shapiro and Taylor characterize the Hilbert-Schmidt composition operators on the Dirichlet space in [29, Proposition 2.4]. The following proposition is proved there.

Proposition G *C_ϕ is a Hilbert-Schmidt operator on $\mathcal{D}$ if and only if*

$$\int_U \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2} dA(z) < \infty.$$

In view of the two propositions above, the Hilbert-Schmidt composition operators on the Dirichlet space are precisely those composition operators that are bounded from the Bloch space to the Dirichlet space. The next result shows that every bounded composition operator from $\mathcal{B}$ to $\mathcal{D}$, and more generally from $\mathcal{B}$ to B_p $(1 < p < \infty)$, is compact.

Proposition 3.9 *Let ϕ be a holomorphic self-map of U.*

1. *If $1 < p < \infty$ then*

$$\int_U \frac{|\phi'(z)|^p(1-|z|^2)^{p-2}}{(1-|\phi(z)|^2)^p} dA(z) < \infty$$

if and only if $C_\phi : \mathcal{B} \to B_p$ is a compact operator (hence $C_\phi : \mathcal{B} \to BMOA$ is a compact operator as well).

2. *If*

$$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2(1-\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z) = 0$$

then $C_\phi : \mathcal{B} \to BMOA$ is a compact operator.

Proof of *(1)***.** Let (f_n) be a bounded sequence in $\mathcal{B}$ such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$. Then,

$$\begin{aligned} \|C_\phi f_n\|_{B_p}^p &= \int_U |f_n'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= \int_{\{z\in U:\delta<|\phi(z)|<1\}} + \int_{\{z\in U:|\phi(z)|\le\delta\}} |f_n'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= I + II \qquad (3.11) \end{aligned}$$

for any $0 < \delta < 1$. Then

$$I \le \|f_n\|_{\mathcal{B}}^p \int_{\{z\in U:1-\delta<|\phi(z)|<1\}} \frac{|\phi'(z)|^p(1-|z|^2)^{p-2}}{(1-|\phi(z)|^2)^p} dA(z)\,, \qquad (3.12)$$

for any $\delta > 0$. Hence, as $\delta \to 0$, $I \to 0$ by the Lebesgue Dominated Convergence Theorem ([27, Theorem 1.34, page 26]) and our hypothesis. Let $\epsilon > 0$ be given. Choose $\delta \in (0,1)$ such that if $h < \delta$ then $I < \epsilon$. For such an h,

$$II = \int_{\{z\in U:|\phi(z)|\le h\}} |\phi'(z)|^p |f_n'(\phi(z))|^p (1-|z|^2)^{p-2} dA(z) < \epsilon \|\phi\|_{B_p}^p \qquad (3.13)$$

for n large enough, since $f_n' \to 0$ uniformly on $\{z \in U : |\phi(z)| \le h\}$. Thus (3.11) and (3.13) imply that there exist a positive integer N such that if $n \ge N$ then $\|C_\phi f_n\|_{B_p} <$ const. ϵ. Thus, $\|C_\phi f_n\|_{B_p} \to 0$, as $n \to \infty$, and Theorem 2.11 yields that $C_\phi : \mathcal{B} \to B_p$ is a compact operator. The converse follows from Proposition 3.8.

Proof of *(2)***.** Let (f_n) be a bounded sequence in $\mathcal{B}$ such that $f_n \to 0$ uniformly on compact sets as $n \to \infty$. Let $\epsilon > 0$ be given. Then by our hypothesis there is a

$\delta > 0$ such that if $|q| > 1 - \delta$ then

$$\int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} < \epsilon\,. \tag{3.14}$$

Fix $q \in U$ such that $|q| > 1 - \delta$. Then

$$\begin{aligned}
&\int_U |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z)\\
&\quad = \int_U |f_n'(\phi(z))|^2(1-|\phi(z)|^2)^2\frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2}dA(z)\\
&\quad \le ||f_n||_{\mathcal{B}}^2\int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2}dA(z)\ \text{(by (3.14))}\\
&\quad \le \text{const.}\,\epsilon\,. \qquad (3.15)
\end{aligned}$$

If $|q| \le 1 - \delta$ then

$$\begin{aligned}
&\int_U |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z)\\
&\quad = \int_U |f_n'(\phi(z))|^2|\phi'(z)|^2\frac{(1-|q|^2)(1-|z|^2)}{|1-\overline{q}z|^2}dA(z) \qquad \text{(by (1.1))}\\
&\quad \le \text{const.}\int_U |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|z|^2)dA(z) \qquad \left(\frac{1-|q|^2}{|1-\overline{q}z|^2} \le \frac{2}{\delta}\right). \quad (3.16)
\end{aligned}$$

Since

$$\int_U \frac{|\phi'(z)|^2(1-|z|^2)}{(1-|\phi(z)|^2)^2} < \infty,$$

$$\lim_{h\to 0}\int_{|\phi(z)|>1-h} \frac{|\phi'(z)|^2(1-|z|^2)}{(1-|\phi(z)|^2)^2}dA(z) = 0\,.$$

Therefore without loss of generality

$$\int_{|\phi(z)|>1-\delta} \frac{|\phi'(z)|^2(1-|z|^2)}{(1-|\phi(z)|^2)^2} < \epsilon. \tag{3.17}$$

Using (3.17) it is easy to see, similarly to the proof of part *(1)*, that

$$\int_U |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|z|^2)dA(z)$$

$$= \int_{|\phi(z)|>1-\delta} + \int_{|\phi(z)|\leq 1-\delta} |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|z|^2)dA(z)$$

$$< \text{const. } \epsilon \tag{3.18}$$

for n large. Then (3.15), (3.16), and (3.18) yield

$$\sup_{q\in U} \int_U |f_n'(\phi(z))|^2|\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z) < \text{const. } \epsilon\ .$$

Therefore by (2.3), $||C_\phi f_n||_* <$ const. ϵ. Thus, $||C_\phi f_n||_* \to 0$ as $n \to \infty$, and Lemma 2.11 implies that $C_\phi : \mathcal{B} \to BMOA$ is a compact operator. This finishes the proof of the proposition. □

In view of Propositions 3.8 and 3.9 we obtain the following corollary.

Corollary 3.10 *For $1 < p < \infty$, every bounded composition operator from $\mathcal{B}$ to B_p is compact.*

Corollary 3.10 also follows from some nontrivial Banach space theory. Here is an outline of the argument. First, B_p is isomorphic to l^p, since the L^p Bergman space of U is isomorphic to l^p (see [15, Theorem 6.2, page 247]). Next $\mathcal{B}$ is isomorphic to l^∞, since $\mathcal{B}$ is isomorphic to the dual of the L^1 Bergman space of U (see [10, Theorem 10 page 49]), which in turn is isomorphic to l^∞. Moreover, if $1 < p < \infty$ then every bounded linear operator $T : l^p \to l^1$ is compact (see [16, Proposition 2.c.3, page 76]). Thus, $T^* : l^\infty \to l^q$ is compact for any $q \in (1, \infty)$. Also a bounded operator is an adjoint if and only if it is weak-star continuous. It is not difficult to show that if $C_\phi : \mathcal{B} \to B_p$ is bounded, then it is weak-star continuous, and hence by the above argument, also compact.

Next we give a characterization of compact composition operators whose range is a subset of $VMOA$.

Theorem 3.11 *Let ϕ be a holomorphic self-map of U, and X a Möbius invariant Banach space. Then $C_\phi : X \to VMOA$ is a compact operator if and only if*

$$\lim_{|q|\to 1} \sup_{\substack{||f||_X<1 \\ f\in X}} \int_U |f'(\phi(z))|^2|\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z) = 0.$$

Proof. First suppose that $C_\phi : X \to VMOA$ is a compact operator. Then $A = \mathrm{cl}\left(\{f\circ\phi \in VMOA : ||f||_X < 1\}\right)$, the $VMOA$ closure of the image under C_ϕ of the unit ball of X, is a compact subset of $VMOA$. Let $\epsilon > 0$ be given. Then there is a finite subset of X, $B = \{f_1, f_2, f_3, ..., f_N\}$, such that each function in A lies at most ϵ distant from B. That is, if $g \in A$ then there exists $j \in J = \{1, 2, 3, ..., N\}$ such that

$$||g - f_j \circ \phi||_* < \frac{\epsilon}{4}. \tag{3.19}$$

Since $\{f_j \circ \phi : j \in J\} \subset VMOA$, there exists a $\delta > 0$ such that for all $j \in J$ and $|q| > 1-\delta$,

$$\int_U |(f_j\circ\phi)'(z)|^2(1-|\alpha_q(z)|^2)dA(z) < \frac{\epsilon}{4}. \tag{3.20}$$

By (3.19) and (3.20) we obtain that for each $|q| > 1-\delta$ and $f \in X$ such that $||f|||_X < 1$ there exists $j \in J$ such that

$$\int_U |(f\circ\phi)'(z)|^2(1-|\alpha_q(z)|^2)dA(z)$$

$$\leq 2\int_U |(f\circ\phi - f_j\circ\phi)'|^2(1-|\alpha_q(z)|^2)dA(z) + 2\int_U |(f_j\circ\phi)'(z)|^2(1-|\alpha_q(z)|^2)dA(z)$$

$$< 2\frac{\epsilon}{4} + 2\frac{\epsilon}{4} = \epsilon.$$

This proves one direction.

In order to prove the converse, let (f_n) be a sequence in the unit ball of X. By Lemma 2.9 and Montel's Theorem there exists a subsequence $n_1 < n_2 < ...$ and a function g holomorphic on U such that $f_{n_k} \to g$ uniformly on compact sets, as $k \to \infty$. By our hypothesis and Fatou's Lemma it is easy to see that $C_\phi g \in VMOA$. We will show that $||C_\phi(f_{n_k} - g)||_* \to 0$, as $k \to \infty$. In order to simplify the notation we will assume, without loss of generality, that we are given a sequence (f_n) in the unit ball of X such that $f_n \to 0$ uniformly on compact sets, as $n \to \infty$. We will show that

$$\lim_{n\to\infty} ||C_\phi f_n||_* = 0. \tag{3.21}$$

To prove (3.21) we will use the equivalent $BMOA$ norm as given by (1.5). Thus, our hypothesis is equivalent to

$$\lim_{h\to 0} \sup_{\substack{\theta\in[0,2\pi) \\ \{f\in X: ||f||_X<1\}}} \frac{1}{h}\int_{S(h,\theta)} |(f\circ\phi)'(z)|^2(1-|z|^2)dA(z) = 0\,. \tag{3.22}$$

Let $\epsilon > 0$ be given. By (3.22), there exists a $\delta > 0$ such that if $n \in N$, $\theta \in [0, 2\pi)$, and $h < \delta$ then

$$\frac{1}{h}\int_{S(h,\theta)} |(f_n\circ\phi)'(z)|^2(1-|z|^2)dA(z) < \epsilon. \tag{3.23}$$

Fix $h_0 < \delta$, $\theta \in [0, 2\pi)$, $n \in N$, and $h \geq \delta$. It is easy to see that there exists $\{\theta_1, \theta_2, ..., \theta_N\} \subset [0, 2\pi)$ such that $S(h, \theta)$ is the union of the sets $\{S(h_0, \theta_j) : j =$

$1, 2, ..., N\}$ and K, a compact subset of U. Hence,

$$\frac{1}{h}\int_{S(h,\theta)} |(f_n \circ \phi)'(z)|^2(1-|z|^2)dA(z)$$

$$\leq \sum_{j=1}^{N} \frac{1}{h_0}\int_{S(h_0,\theta_j)} + \frac{1}{h_0}\int_K |(f_n \circ \phi)'(z)|^2(1-|z|^2)dA(z)$$

$$= I + II. \tag{3.24}$$

Since $f_n' \to 0$ uniformly on K, as $n \to \infty$, there exists an $N \in N$ such that for $n \geq N$

$$II \leq \frac{\epsilon}{h_0}\int_K (1-|z|^2)dA(z) \leq \text{const. } \epsilon. \tag{3.25}$$

Moreover (3.23) yields,

$$I \leq \sum_{j=1}^{N} \epsilon = \text{const. } \epsilon. \tag{3.26}$$

Hence (3.23), (3.24), (3.25), and (3.26) yield (3.21). Thus Lemma 2.11 yields that $C_\phi : X \to VMOA$ is a compact operator. $\square$

There are symbols ϕ such that C_ϕ is compact on $BMOA$ but not on $VMOA$. For example, consider the self-map $\phi(z) = \frac{1}{2}\exp\{\frac{z+1}{z-1}\}$. Since $||\phi||_\infty < 1$, Corollary 2.12 yields that C_ϕ is a compact operator on $BMOA$. Moreover since $\phi \notin \mathcal{B}_0$, C_ϕ is not even bounded on $VMOA$ (Theorem C, page 22).

If $\phi \in VMOA$ then compactness of C_ϕ on $BMOA$ implies the compactness of C_ϕ on $VMOA$. If T is a compact operator on a Banach space X, and Y is an invariant subspace of X such that $T : Y \to Y$ is bounded, then $T : Y \to Y$ is a compact operator as well. Thus we obtain the following proposition.

Proposition 3.12 *Let ϕ be a holomorphic self-map of U. Then,*

1. *If $\phi \in VMOA$ and $C_\phi : BMOA \to BMOA$ is a compact operator then C_ϕ :*

$VMOA \to VMOA$ is a compact operator.

2. *If $\phi \in \mathcal{B}_0$ and $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator then $C_\phi : \mathcal{B}_0 \to \mathcal{B}_0$ is a compact operator.*

Next we show that the sufficient condition of compactness of $C_\phi : \mathcal{B} \to BMOA$ in Proposition 3.9 is also necessary for the compactness of $C_\phi : \mathcal{B} \to VMOA$. We will use Khintchine's inequality for gap series (as done by Arazy, Fisher, and Peetre in [2, Theorem 16]), and Theorem 3.11.

Theorem 3.13 *Let ϕ be a holomorphic self-map of U. Then the following are equivalent:*

1. *$C_\phi : \mathcal{B} \to VMOA$ is a compact operator.*

2. $$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z) = 0\,.$$

Proof. First, suppose that *(1)* holds. Then by Theorem 3.11 and since

$$f_\theta(z) = \sum_{n=0}^{\infty} (e^{i\theta} z)^{2^n} \in \mathcal{B},$$

for all $\theta \in [0, 2\pi)$ (see [1, Lemma 2.1]),

$$\lim_{|q|\to 1} \sup_{\theta\in[0,2\pi)} \int_U \left| \sum_{n=0}^{\infty} 2^n (e^{i\theta} w)^{2^n-1} \right|^2 N(w,q,\phi) dA(w) = 0\,.$$

Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that for any $q \in U$ with $|q| > 1-\delta$ and any $\theta \in [0, 2\pi)$,

$$A_\theta \stackrel{\text{def.}}{=} \int_U \left| \sum_{n=0}^{\infty} 2^n (e^{i\theta} w)^{2^n-1} \right|^2 N(w,q,\phi) dA(w) < \epsilon\,. \tag{3.27}$$

Upon integrating (3.27) with respect to $\frac{d\theta}{2\pi}$ and using Fubini's Theorem, we obtain

$$\int_0^{2\pi} A_\theta \frac{d\theta}{2\pi} = \int_U \left\{ \int_0^{2\pi} \left| \sum_{n=0}^{\infty} 2^n e^{i\theta(2^n-1)} w^{2^n-1} \right|^2 \frac{d\theta}{2\pi} \right\} N(w,q,\phi) dA(w) \leq \epsilon \,. \quad (3.28)$$

Khintchine's inequality (see [36, Theorem V.8.4]) for gap series yields that for any positive integer N

$$\int_0^{2\pi} \left| \sum_{n=0}^{N} 2^n e^{i\theta(2^n-1)} w^{2^n-1} \right|^2 \frac{d\theta}{2\pi} \sim \sum_{n=0}^{N} 2^{2n} |w|^{2^{n+1}-2}. \quad (3.29)$$

Therefore (3.28) and (3.29) imply that

$$\int_0^{2\pi} A_\theta \frac{d\theta}{2\pi} \sim \int_U \left\{ \sum_{n=0}^{\infty} 2^{2n} |w|^{2^{n+1}-2} \right\} N(w,q,\phi) dA(w). \quad (3.30)$$

It is shown in [2, Theorem 16] that

$$\sum_{n=0}^{\infty} 2^{2n} |w|^{2^{n+1}} \geq \frac{\text{const.}}{(1-|w|^2)^2} \,, \quad (3.31)$$

for any $w \in U$ such that $|w| \geq \frac{1}{2}$. Hence (3.28), (3.30), and (3.31) yield

$$\int_U \frac{N(w,q,\phi)}{(1-|w|^2)^2} dA(w) \leq \text{const.} \int_0^{2\pi} A_\theta \frac{d\theta}{2\pi} < \text{const.}\, \epsilon \,, \quad (3.32)$$

for any $q \in U$ with $|q| > 1 - \delta$, and any $\epsilon > 0$. Thus (3.32) yields *(2)*.

Conversely, suppose that *(2)* holds. Fix f in the unit ball of the Bloch space. Then,

$$\int_U |f'(\phi(z))|^2 |\phi'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z)$$

$$\leq ||f||_{\mathcal{B}}^2 \int_U \frac{|\phi'(z)|^2 (1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z)$$

$$\leq \int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z).$$

The righthand side of the above inequality tends to 0, as $|q| \to 1$, by our hypothesis. Hence Theorem 3.11 yields that $C_\phi : \mathcal{B} \to VMOA$ is a compact operator. This finishes the proof of the theorem. □

Proposition 3.14 *Let ϕ be a holomorphic self-map of U. If $C_\phi : BMOA \to VMOA$ is a compact operator then*

$$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{1-|\phi(z)|^2} dA(z) = 0\,.$$

Proof. By Theorem 3.11 and since $f_\theta(z) = \log \frac{1}{1-e^{-i\theta}z} \in BMOA$ for all $\theta \in [0, 2\pi)$,

$$\lim_{|q|\to 1} \sup_{\theta\in[0,2\pi)} \int_U |f'_\theta(z)|^2 N(w,q,\phi) dA(z) = 0\,. \tag{3.33}$$

Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that for any $q \in U$ with $|q| > 1-\delta$ and any $\theta \in [0, 2\pi)$,

$$\begin{aligned} A_\theta &= \int_U |f'_\theta(z)|^2 N(w,q,\phi) dA(z) \\ &= \int_U \frac{1}{|1-e^{-i\theta}w|^2} N(w,q,\phi) dA(w) < \epsilon \end{aligned} \tag{3.34}$$

Integrating (3.34) with respect to $\frac{d\theta}{2\pi}$ and Fubini's Theorem yield

$$\int_0^{2\pi} A_\theta \frac{d\theta}{2\pi} = \int_U \left\{ \int_0^{2\pi} \frac{1}{|1-e^{-i\theta}w|^2} \frac{d\theta}{2\pi} \right\} N(w,q,\phi) dA(w) \leq \epsilon\,.$$

Thus,

$$\int_U \frac{N(w,q,\phi)}{1-|w|^2} dA(w) < \epsilon \,,$$

for all $|q| > 1-\delta$, and all $\epsilon > 0$. Therefore

$$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{1-|\phi(z)|^2} dA(z) = 0 \,.$$

□

Next we show that composition operators on $BMOA$ and $VMOA$, where the symbol is a boundedly valent holomorphic function whose image lies inside a polygon inscribed in the unit circle, are compact if and only if they are compact on the Bloch space. We will use Propositions 3.9, 3.12, and the following theorem of Pommerenke ([24, Satz 1]).

Theorem H *Let f be a holomorphic function on U such that*

$$\sup_{w_0} \int_{|w-w_0|<1} \eta(f,w) dA(w) < \infty \,,$$

where the supremum is extended over all points w_0 in the complex plane. Then,

$$f \in BMOA \Leftrightarrow f \in \mathcal{B}, \;\; f \in VMOA \Leftrightarrow f \in \mathcal{B}_0 \,.$$

In page 46 we defined a nontangential approach region Ω_α $(0 < \alpha < 1)$ in U with vertex $\zeta \in \partial U$. The exact shape of the region is not relevant. The important fact that we will use in the theorem below is that there exists $0 < r < 1$ and $c > 0$ such that if $z \in \Omega_\alpha$ and $|\zeta - z| < r$, then

$$|\zeta - z| \leq c(1-|z|^2). \tag{3.35}$$

Theorem 3.15 *Let ϕ be a boundedly valent holomorphic self-map of U such that $\phi(U)$ lies inside a polygon inscribed in the unit circle. Then the following are equivalent:*

1. $C_\phi : \mathcal{B} \to VMOA$ *is a compact operator.*
2. $C_\phi : \mathcal{B} \to BMOA$ *is a compact operator.*
3. $C_\phi : BMOA \to BMOA$ *is a compact operator.*
4. $C_\phi : \mathcal{B} \to \mathcal{B}$ *is a compact operator.*
5. $C_\phi : \mathcal{B}_0 \to \mathcal{B}_0$ *is a compact operator.*
6. $C_\phi : VMOA \to VMOA$ *is a compact operator.*

Proof. *(1)⇒(2)⇒(3)* is clear.

(3)⇒*(4)*. This is valid for all holomorphic self-maps of U (Proposition 3.2).

(4)⇒(5). Since ϕ is a boundedly valent holomorphic self-map of U, $\phi \in \mathcal{D} \subset VMOA \subset \mathcal{B}_0$. Thus $\phi \in \mathcal{B}_0$. The compactness of C_ϕ now follows from part *(2)* of Proposition 3.12.

(5)⇒*(1)*. By Madigan and Matheson's Theorem 1 (see [22]) C_ϕ is a compact operator on the little Bloch space if and only if

$$\lim_{|z|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2} = 0\,.$$

It follows that $\log \frac{1}{w-\phi} \in \mathcal{B}_0$ for each $w \in \partial U$. By Theorem H each boundedly valent function in $\mathcal{B}_0$ must belong to $VMOA$, hence $\log \frac{1}{w-\phi} \in VMOA$ for each $w \in \partial U$. Thus

$$\lim_{|q|\to 1} \int_U \left| \left(\log \frac{1}{w-\phi(z)} \right)' \right|^2 (1-|\alpha_q(z)|^2) dA(z) = 0$$

hence

$$\lim_{|q|\to 1}\int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{|w-\phi(z)|^2}dA(z)=0, \tag{3.36}$$

for each $w \in \partial U$.

Let $\{w_j : 1 \le j \le n\}$ be the vertices of the inscribed polygon containing $\phi(U)$. Break the unit disc up into a compact set K and finitely many regions

$$E_j = \{z \in U : |w_j - \phi(z)| < r\}$$

where r is chosen so that the regions are disjoint, and so that

$$|w_j - \phi(z)| \le \text{const.}(1-|\phi(z)|^2)$$

for each $z \in E_j$ and each j. Then for each $q \in U$,

$$\int_{E_j}\frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2}dA(z) \le \text{const.}\int_{E_j}\frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{|w_j-\phi(z)|^2}dA(z).$$

Hence

$$\begin{aligned}
&\int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2}dA(z)\\
&= \sum_{j=1}^n \int_{E_j} + \int_K \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2}dA(z)\\
&\le \text{const.}\sum_{j=1}^n \int_{E_j}\frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{|w_j-\phi(z)|^2}dA(z)\\
&+ \text{const.}\int_U |\phi'(z)|^2(1-|\alpha_q(z)|^2)dA(z)\,,
\end{aligned} \tag{3.37}$$

for all $q \in U$.

Any boundedly valent holomorphic self-map of U belongs to $VMOA$. Hence (3.36) and (3.37) imply that

$$\lim_{|q|\to 1} \int_U \frac{|\phi'(z)|^2(1-|\alpha_q(z)|^2)}{(1-|\phi(z)|^2)^2} dA(z) = 0. \tag{3.38}$$

By (3.38) and Theorem 3.13 we obtain that $C_\phi : \mathcal{B} \to VMOA$ is a compact operator.

Proposition 3.12 yields *(3)* $\Rightarrow$ *(6)*. If *(6)* holds, that is $C_\phi : VMOA \to VMOA$ is a compact operator, then C_ϕ is weakly compact on $VMOA$. Hence by Theorem VI 5.5 in [9, page 189], $C_\phi(BMOA) \subset VMOA$. Thus $\log \frac{1}{w-\phi(z)} \in VMOA$ $(w \in \partial U)$. Thus by the proof of *(5)* $\Rightarrow$ *(1)* we obtain *(6)* $\Rightarrow$ *(1)*. This finishes the proof of the theorem. □

Definition 3.16 *A region $G \subset U$ is said to have a nontangential cusp at $\zeta \in \partial U$ if*

$$\lim_{\substack{z\to 1\\ z\in G}} \frac{|\mathrm{Im} z|}{|1-z|} = 0\,.$$

Note Theorem 3.5, Theorem 3.15 and Madigan and Matheson's Theorem 5 (see [22, page 2685]) yield that if ϕ is a univalent self-map of U such that $\phi(U)$ has finitely many points of contact with ∂U and such that at each of these points $\phi(U)$ has a nontangential cusp, then C_ϕ is a compact operator on B_p $(p > 2)$, on $BMOA$, and on $VMOA$.

CHAPTER 4

Final remarks and questions

Madigan and Matheson showed that if $C_\phi : \mathcal{B}_0 \to \mathcal{B}_0$ is weakly compact then it is compact. Is a similar statement valid for $C_\phi : VMOA \to VMOA$? That is, does $C_\phi(BMOA) \subset VMOA$ imply that C_ϕ is a compact operator on $VMOA$?

In Theorem 3.15 we showed that for certain boundedly valent holomorphic self-maps of U, compactness of C_ϕ on $BMOA$ is equivalent to the compactness of C_ϕ on $\mathcal{B}$. Is this true for all boundedly valent symbols?

In Theorem 3.15 we used that ϕ is boundedly valent to be able to conclude that if $\log \frac{1}{w-\phi(z)} \in \mathcal{B}_0$ then $\log \frac{1}{w-\phi(z)} \in VMOA$ $(w \in \partial U)$. We should mention here that Stroethoff, using an area version of the $BMOA$ counting functions, characterizes exactly when a function $\phi \in \mathcal{B}_0$ belongs to $VMOA$. He showed in [33, page 78] that a function $\phi \in \mathcal{B}_0$ belongs to $VMOA$ if and only if for every $\delta > 0$

$$\lim_{|p|\to 1} \sup_{\substack{w \\ |\phi(p)-w|\geq\delta}} \int_0^1 t\eta(\phi\circ\alpha_p - w, t)dt = 0.$$

In Theorem 3.13 we showed that the compactness of $C_\phi : \mathcal{B} \to VMOA$ is determined by the "behavior" of $\{C_\phi \sum \left(e^{i\theta}\phi(z))^{2^n}\right) : \theta \in [0, 2\pi)\}$. Does a similar statement hold for compact operators $C_\phi : BMOA \to VMOA$? That is, is it true that $C_\phi : BMOA \to VMOA$ is a compact operator if and only if

$$\lim_{|q|\to 1} \sup_{\theta\in[0,2\pi)} \int_U \left| \left(\log \frac{1}{1-e^{i\theta}\phi(z)} \right)' (z) \right|^2 (1-|\alpha_q(z)|^2) dA(z) = 0\,?$$

BIBLIOGRAPHY

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[1] J.M. Anderson, J. Clunie and Ch. Pommerenke, "On Bloch functions and normal functions," Journal für die Reine und Angewandte Mathematik **270** (1974), 12-37.

[2] J. Arazy, S.D. Fisher and J. Peetre, "Mobius invariant function spaces," Journal für die Reine und Angewandte Mathematik **363** (1985), 110-145.

[3] A. Baernstein, "Univalence and bounded mean oscillation," Michigan Mathematics Journal, **23** (1976), 217-223.

[4] A. Baernstein, "Analytic functions of bounded mean oscillations," pages 3-36 in *Aspects of Contemporary Complex Analysis,* edited by D.A. Brannan and J.G. Clunie, Academic Press, London, 1980.

[5] P.S. Chee, "VMOA and vanishing Carleson measures," Complex Variables, **25** (1994), 311-322.

[6] J.A. Cima and G. Schober, "Analytic functions with bounded mean oscillation and logarithms of H^p -functions," Math. Z., **151** (1976), 295-300.

[7] J. Cima and W. Wogen, "A Carleson theorem for the Bergman space on the ball," Journal of Operator Theory, **7** (1982), 157-165.

[8] J.B. Conway, *Functions of one Complex Variable*, second edition, Springer-Verlag, New York, 1978.

[9] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.

[10] P.L. Duren, B.W. Romberg, A.L. Shields, "Linear functionals on H^p spaces with $0 < p < 1$," Journal für die Reine und Angewandte Mathematik, **238** (1969), 32-60.

[11] P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[12] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[13] F. Holland and D. Walsh, "Growth estimates for functions in the Besov space A_p," Proceedings of the Royal Irish Academy, **Vol. 88A No. 1** (1988), 1-18.

[14] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publications, Inc., New York, 1976.

[15] J. Lindenstrauss and A. Pelczynski, "Contributions to the Theory of the Classical Banach spaces," Journal of Functional Analysis **8** (1971), 225-249.

[16] J. Lindenstrauss and A. Tzafriri, *Classical Banach Spaces, Volume I*, Springer-Verlag, New york, 1977.

[17] D.H. Luecking, "A technique for characterizing Carleson measures on Bergman spaces," Proceedings of the American Mathematical Society **87** (1983), 656-660.

[18] D.H. Luecking, "Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives," American Journal of Mathematics **107** (1985), 85-111.

[19] B.D. MacCluer and J.H. Shapiro, " Angular derivatives and compact composition operators on the Hardy and Bergman spaces," Canadian Journal of Mathematics **38** (1986), 878-906.

[20] B.D. MacCluer, "Composition operators on S^p," Houston Journal of Mathematics **13** (1987), 245-254.

[21] K.M. Madigan, "Composition operators on analytic Lipschitz spaces," Proceedings of the American Mathematical Society **119** (1993), 465-473.

[22] P.K. Madigan and A. Matheson, "Compact composition operators on the Bloch space," Transactions of the American Mathematical Society **347** (1995), 2679-2687.

[23] V.V. Peller "Hankel operators of class τ_p and their applications. (Rational approximation, Gaussian Processes, the problem of majorising operators,)" English translation: Math. USSR Sbornik **Vol. 41**, **No. 4** (1980), 443-479.

[24] C. Pommerenke, "Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation," Commentarii Mathematici Helvetici **52** (1977), 5911-602.

[25] R.C. Roan, "Composition operators on a space of Lipschitz functions," Rocky Mountain Journal of Mathematics **10** (1980), 371-379.

[26] L.A. Rubel and R.M. Timoney, "An extremal property of the Bloch space," Proceedings of the American Mathematical Society **75** (1979), 45-49.

[27] W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill, New York, 1987.

[28] D. Sarason, *Operator Theory on the Unit Circle*, Lecture notes of Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1978.

[29] J.H. Shapiro and P.D. Taylor, “Compact, nuclear and Hilbert-Schmidt composition operators on H^2, Indiana University Mathematics Journal **125** (1973), 471-496.

[30] J.H. Shapiro, “ Compact composition operators on spaces of boundary-regular holomorphic functions,” Proceedings of the American Mathematical Society **100** (1987), 375-404.

[31] J.H. Shapiro, “The essential norm of a composition operator,” Annals of mathematics **125** (1987), 375-404.

[32] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[33] K.M.R. Stroethoff, “Characterizations of the Bloch space and related spaces,” Thesis, Michigan State University, 1987.

[34] K. Zhu, *Operator Theory on Function Spaces*, Marcel Dekker, New York, 1990.

[35] K. Zhu, “Analytic Besov spaces,” Journal of Mathematical Analysis and Applications **157** (1991), 318-336.

[36] A. Zygmund, *Trigonometric Series*, Volume I, second edition, Cambridge University Press, Cambridge, 1959.